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THE MECHANICAL DESIGN AND CONSTRUCTION OF COMMUTATORS.

By R. LIVINGSTONE.

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THE MECHANICAL DESIGN AND CONSTRUCTION OF COMMUTATORS.

BY

R. LIVINGSTONE.

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PREFACE

THIS book is intended principally for draughtsmen and engineers engaged in the design of electrical machinery, and it is hoped that it will also be found useful by students as an example of the application of mechanics to practical design. No attempt has been made to deal with any of the electrical problems of commutation, as this work has already been undertaker by more competent writers. A knowledge of elementary mechanics on the part of the reader has been assumed and also a knowledge of mathematics. The calculus has been used as little as possible, and only in cases where other methods would be tedious and uninteresting. It is impossible in a work of this description to deal with the design of every type of commutator in use, but the two types dealt with represent the most general present-day practice in commutator construction, and will serve to show the principles involved in design. With the aid of a little ingenuity the reader will be able to apply the same principles to any modification of the two types. The working of most of the formulæ has been given fully, so that no difficulty should be found in following the reasoning.

As so little has been published on the design of commutators much of the work in this book is entirely

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original, and it is possible that some engineers may not quite agree with the conclusions arrived at. The reasoning in every case has been given fully, however, and it is hoped that this work will be found of some assistance to designers of electrical machinery. While every effort has been made to avoid errors, the author would be glad to have any that have escaped notice pointed out to him.

R. LIVINGSTONE.

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CHAPTER I.

INTRODUCTORY.

Considering the importance of the mechanical design of a commutator, it is surprising that so little has been published bearing directly on this subject. Engineers seem to have confined themselves to the problems of the electrical design and the process of manufacture, and to have left the mechanical design severely alone. By mechanical design I do *not* mean descriptions of current practice.

A commutator to be entirely satisfactory should:—

1st. Have an ample wearing depth of copper, and be so designed that when worn down to the limit the stresses in the materials will not be unduly increased.

2nd. Be built together so rigidly that a blow on the surface with a light hammer will not displace any bar.

3rd. Be so designed that the strains due to initial tension, expansion and centrifugal force will be well within the elastic limits of the materials, so that there will be no permanent set and consequent loosening of the bars on cooling down.

4th. Have mica of such a quality and thickness between bars that the copper and the mica will wear uniformly.

As regards the first requirement, it is customary to provide from $\frac{3}{4}$ in. radial wearing depth on small machines to $1\frac{1}{2}$ in. on large machines. Judging from the wear on well-kept commutators, this allowance is certainly ample. If every commutator, however, is designed to last the same length of time, and the amount of wear on a bar due to sparking and friction

every time it passes under a brush is a constant, then the wearing depth should be

$$= cNn, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

when N = revolutions per minute,

n = number of brush spindles,

and c = a constant depending on the desired life of the commutator.

For ordinary generators a value of $c = 0{\cdot}0005$ to $0{\cdot}00055$ may be taken when getting out a design for a new machine. Of course such a formula as the preceding, although of interest, is of very little use, as the consulting engineer usually specifies the allowance for wear which must be made. The type of construction shown in Fig. 1 seems to be the type which best fulfils the second portion of the first requirement.

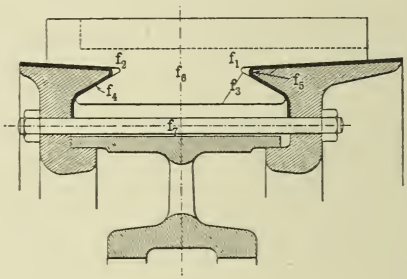


FIG. 1.

In order that the second requirement may be satisfied, it is necessary to design the commutator so that a tension to 12,000 lb. per square inch in the bolts will give a sufficiently large tangential pressure on the mica between the bars.

In assembling a commutator the workman must be expected to exercise a certain amount of discrimination and may be expected to tighten up the bolts in the clamping ring to a stress of approximately 12,000 lb. per square inch. Advantage will be taken of this to proportion the number and diameter of the bolts to the diameter of the commutator and the area of the bar.

Of course, a much greater stress than 12,000 lb. per square inch is often put on bolts, but we must treat the commutator

bolts with the care which such an important portion of the machine deserves.

It must not be supposed that it is possible to tighten up the commutator sufficiently by the bolts in the end clamps alone. There is such a large amount of friction between the micanite end ring and the copper bars that this is hardly practicable. The commutator must be pressed together to the required degree of rigidity in the assembling jig, and then the bolts should be tightened up so as to *just* take the pressure exerted by the jig, without stretching when the jig is released. Too great a tension on the bolts will only overstrain the commutator, and too little tension might allow the commutator to "give" when the jig is released and so become less rigid than it ought to be. It is evident that this point is of primary importance in commutator construction.

The third requirement is of the utmost importance, and it is only by very careful design of every portion of the commutator, combined with the greatest care in its construction, that it is possible to ensure that it will be fulfilled.

The methods of computing the stress in the various parts of the commutator will be fully dealt with in a later Chapter.

With regard to the last requirement, the thickness of mica employed depends to a considerable extent on the quality of the mica, the degree of hardness of the commutator bars, and to some extent on the diameter of the commutator.

In commutators of small diameter, with hard-drawn copper bars, an ordinary quality of mica $\frac{1}{32}$ in. thick will be found to wear at the same rate as the copper bars, but if soft amber mica is used it should be about 0·05 in. thick. Thinner mica segments than this will be found to wear at a greater rate than the copper, forming receptacles between the bars, in which the copper and carbon dust will accumulate and bridge over adjacent commutator bars.

In commutators of large diameter, however, the mica used should be somewhat thinner than that found suitable for small commutators. With hard-drawn copper bars and soft amber mica segments the mica should be approximately $\frac{1}{32}$ in. or 0·03 in. thick, but with the ordinary quality of mica the

thickness should not exceed 0·025 in. Thicker mica than this is found to wear at a less rate than the copper, forming slight ridges between the bars and so causing sparking.

Stresses in Commutators.—The causes of stress in the different parts of a commutator are centrifugal force, initial tension, and expansion due to heating. In the following investigations I will obtain expressions for the stress at different points, taking the causes in the above order.

Judging from the published articles on the mechanical design of various machines it does not appear to have been previously recognised that the stress due to bending in such a cantilever as that shown in Fig. 4 (p. 11) has its maximum value at some angle from the perpendicular, depending on the ratio D/L. That this is so is evident from the mathematical proof given. The angle given, however, is only strictly true within the elastic limits of the material, and depends on the modulus of elasticity in tension being equal to the modulus of elasticity in compression. The curve giving the angle of maximum stress will be found useful in the design of many portions of machines—for example, the head of a T bolt—and it will be found that the maximum stress in cantilevers when the ratio D/L is large is considerably greater than the stress in the perpendicular section. In dealing with expansion stresses many short cuts have been purposely avoided. It is much better to use the longer methods until experience suggests short cuts. Then the reader will understand exactly what work is being avoided by using the abridged method, and will not be in any danger of using such methods where their use would involve serious errors.

For example, it will be found that, in commutators with a particular area of bar, and a certain ratio of thickness of bar to thickness of mica between bars, the bolt stress due to expansion will be the same as the stress obtained by ignoring circumferential expansion entirely and simply computing the bolt stress due to the difference of longitudinal expansion of the bar and the expansion of the bolts. This method, however, is only applicable to a few cases, usually longer commutators than the average, but will indicate how abridgements can be made when sufficient experience has been gained.

From what little has been published on the mechanical design of commutators it is evident that the stresses due to centrifugal force have been regarded as of primary importance, and no importance attached to initial tension and expansion. That the centrifugal stresses are really of small value compared with the stresses due to initial tension and expansion is evident from the numerical values in the example following.

The equations for expansion stresses must not be expected to give mathematically accurate results. They will be found, however, to give values close to the actual stresses, but usually higher than the actual stresses. This is due to neglecting the radial deflection of the copper bar itself. Such deflections, however, reduce the stress by such a small amount that they may be ignored. Indeed, if any attempt were made to include them in the equations found for expansion stresses, the expressions would become so unwieldy that they would be unfit for practical use.

The author has had many opportunities of applying the formulæ to cases in which fracture has taken place due to excessive rise in temperature, and has found that the temperature which should cause such fractures, as given by the equations, agreed very closely with the actual temperatures recorded. In some cases, where the copper below the "V" had given way, the angle of fracture agreed almost exactly with the calculated angle. Such close results, however, have not been observed in materials such as cast iron. This is due to the fact that the instantaneous modulus of elasticity has a different value in tension, at the ultimate strength of the material, from what it has in compression, and in no way shows that the angle given by the curve is not true *within the elastic limit of the material.*

CHAPTER II.

DESIGN OF "V" RING COMMUTATORS.

Centrifugal Stresses.—In Fig. 1 the letters f_1, f_2, &c., indicate the sections which are under highest stress. In most cases it will be sufficiently accurate to regard the section as solid material 1 in. thick, the taper on the bars being usually extremely small. Again, as the centrifugal force and the modulus of the section both vary directly as the thickness of the bar, taking a section 1 in. thick will practically give the same stresses as taking the actual thickness of the bar.

The centrifugal pull of 1 lb. of any material revolving at N revs. per min. and at a radius of gyration of ρ inches is $= 0{\cdot}00002838\rho N^2$.

Then, as copper weighs 0·32 lb. per cubic inch, the centrifugal pull of 1 cubic in. of copper will be $0{\cdot}00000908\rho N^2$. Let this = P.

To obtain the centrifugal pull of any portion of the section under consideration we have only to multiply the area of that particular portion in square inches by P.

Stress at f_1 (Fig. 2).—To obtain the bending moment at this point we may regard the figure as being split up into two separate portions—a parallelogram of depth x and length l, and a triangle whose base is $l \tan \theta$ and whose height is l.

The total B.M. then is P times the area of the parallelogram × the distance of its centre of gravity from f_1 + P times the area of the triangle × the distance of its C.G. from f_1. The area of the parallelogram is xl, and the distance of its C.G. from f_1 is evidently $l/2$.

The area of the triangle is $l \tan \theta \times l/2 = \frac{l^2 \tan \theta}{2}$, and the distance of its C.G. from f_1 is $l/3$. (*See* Appendix.)

The total B.M. then is

$$P\left\{\left(xl \times \frac{l}{2}\right) + \left(\frac{l^2 \tan\theta}{2} \times \frac{l}{3}\right)\right\}$$

$$= P\left\{\frac{xl^2}{2} + \frac{l^3 \tan\theta}{6}\right\}.$$

Now the modulus of the section

$$Z = \frac{(x + l\tan\theta)^2}{6}.$$

$\therefore$ stress at $$f_1 = \frac{Pl^2(3x + l\tan\theta)}{(x + l\tan\theta)^2} \text{ lb. per sq. in.} \quad (2)$$

Now, as x becomes less, due to the wear of the bars, the stress at f_1 will gradually increase to a maximum value and then begin to decrease.

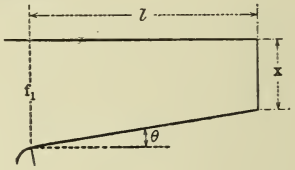


FIG. 2.

The point where the stress is a maximum can be found by plotting the values of the stress for different values of x, or in a much simpler way, involving the calculus, as under:—

$$\frac{df_1}{dx} = \frac{3Pl^2(x + l\tan\theta)^2 - 2Pl^2(x + l\tan\theta)(3x + l\tan\theta)}{(x + l\tan\theta)^4}.$$

Now for a maximum value of f_1, $\frac{df_1}{dx} = 0$;

that is,

$$3Pl^2(x + l\tan\theta)^2 = 2Pl^2(x + l\tan\theta)(3x + l\tan\theta),$$

$$3(x + l\tan\theta) = 2(3x + l\tan\theta),$$

$$l\tan\theta = 3x.$$

$$\therefore \quad x = \frac{l\tan\theta}{3} \quad \ldots\ldots\ldots \quad (3)$$

Substituting this value of x in equation (2), we obtain for maximum stress at

$$f_1 = \frac{9Pl}{8\tan\theta} \text{ lb. per square inch} \quad (4)$$

Now the deflection of this portion of the bar at the outer end is another point we must consider, as, apart from the strength of the material, the overhanging copper must be *stiff*.

For small angles (θ) the following will give the deflection with sufficient accuracy :—

$$\delta = \text{deflection in inches} = \frac{fl^2}{2E\left(\frac{x + l\tan\theta}{2}\right)} \text{ (*see* Appendix), (5)}$$

when E = Young's modulus of elasticity for copper.

Stress at f_2 (Fig. 3).—In this case we will regard the figure as being split up into three separate portions—a parallelogram of

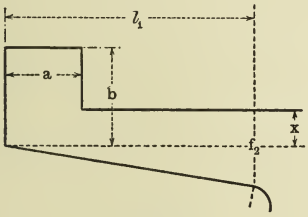


Fig. 3.

depth b and length a, whose C.G. is $(l_1 - a/2)$ from f_2; a parallelogram whose depth is x, whose length is $l_1 - a$, and whose C.G. is $\frac{l_1 - a}{2}$ from f_2; and a triangle of base$= l_1 \tan\theta$, height l_1, and whose C.G. is $l_1/3$ from f_2. (*See* Appendix.)

Then the total B.M.

$$= P\left\{ab\left(l_1 - \frac{a}{2}\right) + l_1\tan\theta \times \frac{l_1}{2} \times \frac{l_1}{3} + x(l_1 - a)\frac{l_1 - a}{2}\right\}$$

$$= P\left\{ab\left(l_1 - \frac{a}{2}\right) + \frac{l_1^3\tan\theta}{6} + \frac{x(l_1 - a)^2}{2}\right\}$$

$$Z = \frac{(l_1\tan\theta + x_1)^2}{6}.$$

$$\therefore \text{stress at } f_2 = \frac{6P\left\{ab\left(l_1 - \frac{a}{2}\right) + \frac{l_1^3\tan\theta}{6} + \frac{x_1(l_1 - a)^2}{2}\right\}}{(l_1\tan\theta + x_1)^2}$$

lb. per square inch (6)

This stress also will increase as the commutator wears down and the section where the stress is a maximum may be found as follows:—

In equation (6) let $ab\left(l_1 - \frac{a}{2}\right) + \frac{l_1^3 \tan\theta}{6} = C$; then

$$\frac{df_2}{dx_1} = \frac{6P\left\{(l_1\tan\theta + x_1)^2\frac{(l_1-a)^2}{2} - \left(C + \frac{(l_1-a)^2}{2}x_1\right)2(l_1\tan\theta + x_1)\right\}}{(l_1\tan\theta + x_1)^4}.$$

Now, for maximum value of f_2, $\frac{df_2}{dx_1} = 0$;

that is,

$$(l_1\tan\theta + x_1)^2\frac{(l_1-a)^2}{2} = \left(C + \frac{(l_1-a)^2}{2}x_1\right)2(l_1\tan\theta + x_1)$$

$$= (l_1\tan\theta + x_1)\frac{(l_1-a)^2}{2} = 2\left(C + \frac{x_1(l_1-a)^2}{2}\right),$$

$$l_1\tan\theta + x_1 = \frac{2\{2C + x_1(l_1-a)^2\}}{(l_1-a)^2},$$

$$l_1\tan\theta + x_1 = 2\left\{\frac{2C}{(l_1-a)^2} + x_1\right\},$$

$$l_1\tan\theta - \frac{4C}{(l_1-a)^2} = x_1.$$

Substituting for C its value

$$x_1 = l_1\tan\theta - \frac{4ab\left(l_1 - \frac{a}{2}\right) + \frac{2}{3}l_1^3\tan\theta}{(l_1-a)^2}. \quad . \quad . \quad . \quad (7)$$

In most cases, however, the stress will increase right up to $x_1 = 0$, and in such a case f_2 must be computed with a value for x_1 determined by the limit of wear.

Stress at f_3.—Before going into the stress at this point I will briefly investigate the stress in the form of cantilever shown in Fig. 4. It is a customary mistake to calculate the stress in this form of cantilever in the section X – X. Now, this is *not* the section where the stress has a maximum value; the section where the stress is a maximum is at an angle α depending on the ratio D/L.

In Fig. 4 $\text{B.M.} = W\left(L + \frac{D}{2}\tan\alpha\right)$,

$$Z = \frac{D^2}{6\cos^2\alpha},$$

and
$$f = \frac{6W\cos^2\alpha\left(L + \frac{D}{2}\tan\alpha\right)}{D^2}.$$

Now, $\frac{df}{d\alpha} = -2\cos\alpha\sin\alpha\left(L + \frac{D}{2}\tan\alpha\right) + \cos^2\alpha\left(\frac{D}{2}\sec^2\alpha\right)$,

and for a maximum value of f, $\frac{df}{d\alpha} = 0$.

$$\therefore \quad \cos\alpha\left(\frac{D}{2}\sec^2\alpha\right) = 2\sin\alpha\left(L + \frac{D}{2}\tan\alpha\right)$$

$$= \frac{D}{2}\sec^2\alpha = 2\tan\alpha\left(L + \frac{D}{2}\tan\alpha\right)$$

$$= \frac{D}{2}(\sec^2\alpha - 2\tan^2\alpha) = 2\tan\alpha L$$

$$= \frac{D}{L} = \frac{4\tan\alpha}{\sec^2\alpha - 2\tan^2\alpha} = \frac{4}{\sec^2\alpha\cot\alpha - 2\tan\alpha}. \quad . \quad (8)$$

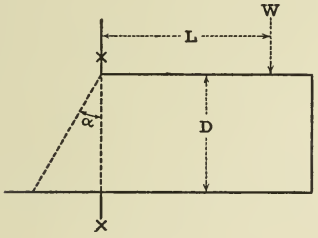


FIG. 4.

By plotting the ratios D/L as ordinates and angles α as abscissæ we obtain the curve shown in Fig. 5. This curve will be found very useful in such a case as the "V" portion of a commutator. I would point out that, although the maximum stresses at f_1 and f_2 are slightly greater than the values given by equations 4 and 6, the angle α in these cases is so small that the difference is negligible. This is not so,

however, when the ratio D/L is so large as it is at f_3. It will also be noticed from equation (8) that the limit of the angle α is 45 deg.

The stress at f_3 can now be easily obtained.

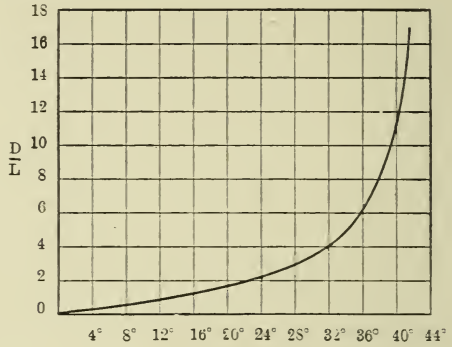


FIG. 5.

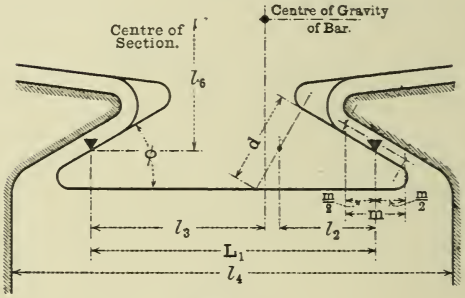


FIG. 6.

Let B = area of whole bar, then from Fig. 6 the load on the front "V" $= \frac{PB}{L_1} l_3$

$$\text{B.M.} = \frac{PBl_3l_2}{L_1}, \quad Z = \frac{d^2}{6}.$$

$\therefore$ stress at $$f_3 = \frac{6PBl_3l_2}{L_1d^2} \text{ lb. per square inch} \quad . \quad . \quad (9)$$

The pressure on the mica "V" ring per square inch will be

$$f_4 = \frac{PBl_3}{L_1m} \text{ lb. per square inch} \quad . \quad . \quad (10)$$

Stress at f_5 (see Fig. 1).—To obtain the stress at this point we must obtain the total volume of copper in the commutator. Let this volume = V, then the load on end ring will be PVl_3/L_1.

In Fig. 7 we will consider the stress with regard to the axis Y – Y. Then B.M. $= \frac{PVl_3l_5}{L_1}$. Now the moment of inertia of this portion of the "V" is $I_1 + 2\pi I_2$, where I_1 = moment of

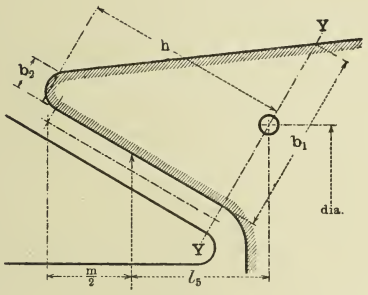


FIG. 7.

inertia of section Y – Y about the axis ⊙ and I_2 = moment of inertia of "V" about axis Y – Y.

Now,
$$I_1 = \frac{b_1^3 \pi \text{ dia.}}{12},$$

and
$$I_2 = \frac{h^3(b_1 + 3b_2)}{12}. \quad (\textit{See} \text{ Appendix.})$$

Then the moment of inertia of whole "V" when in the form of a continuous ring is
$$\frac{\pi\{b_1^3 \text{ dia.} + 2h^3(b_1 + 3b_2)\}}{12}.$$

Then
$$Z = \frac{b_1^3 \text{ dia.} + 2h^3(b_1 + 3b_2)}{3{\cdot}82h},$$

and the stress at $f_5 = \frac{3{\cdot}82h(PVl_3l_5)}{L_1\{b_1^3 \text{ dia.} + 2h^3(b_1 + 3b_2)\}}$ lb. per sq. in. (11)

To this must be added the stress due to the horizontal component of the centrifugal forces tending to bend the end ring as a whole. In this case we must regard the end ring as a uniform beam supported on an *infinite* number of supports,

spaced the pitch of the bolts and loaded with a uniformly distributed load. The horizontal component of the centrifugal pull of the copper will be

$$\frac{PV}{2\cot\phi},$$

and if d = the diameter of the bolt circle, the load per inch on the end ring will be $\frac{PV}{2\pi d\cot\phi}$.

Let the distance between the centres of the bolts $= l$,

and $$\frac{PV}{2\pi d\cot\phi} = W.$$

Then the maximum bending moment between the bolts will be $Wl^2/24$ and the bending moment immediately over the supports $Wl^2/12$.

It will be easier to obtain the moment of inertia of the end ring section by a graphical method, as it is difficult to obtain a convenient mathematical expression for this form of section. A convenient method will be found in most text books on mechanics. From the inertia, the modulus of the section (Z) can easily be obtained by dividing by the distance from the extreme point of the "V" to the neutral axis.

As the stress due to bending of the end ring is only of positive value, with regard to the other stresses, over the supports, we may neglect the bending of the end ring between the bolts. The stress in the end ring, then, will be

$$\frac{Wl^2}{12Z} = \frac{PV}{75{\cdot}5d\cot\phi Z}.$$

Stress at f_6.—This, again, is treated as a beam supported as in Fig. 6 and loaded directly as the area of the bar. This method of treatment can be found in any text book on mechanics. In addition to the B.M. due to the outward pull of the copper, there is a horizontal component due to this force, and this must be added. The value of the B.M. due to the horizontal component is

$$\frac{PBl_6}{2\cot\phi}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In short commutators the tension on the section between the

"V's" might be high, and this also should be computed, keeping in mind the eccentric loading of the section. (*See* Appendix.)

Bolt Stress f_7.—Here we must again take the whole volume of copper.

$$\text{Load on bolts} = \frac{PV}{2 \cot \phi}.$$

Let the total area of bolts at the bottom of the thread = A, then

$$f_7 = \frac{PV}{2 \cot \phi A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This completes the investigation of the stresses due to centrifugal force, and, before going into the stresses due to initial tension and expansion, I would point out that, as all the stresses are directly proportional to the bolt stress, it will not be necessary to go through every portion of the commutator again.

Stresses due to Initial Tension.—In estimating the stresses due to initial tension we may assume that the bolts will be stressed to a tension of 12,000 lb. per square inch when the commutator is being assembled, and that this tension will just balance the forces exerted by the clamping ring. Now the pressure exerted by the clamping ring should be so high that no individual bar can be displaced when the commutator is struck on the surface by a light hammer. The minimum pressure to ensure that the commutator will stand this test can only be ascertained from experiments. For ordinary commutators about 30 in. to 40 in. diameter a tangential pressure of about 900 lb. per square inch of bar will be found sufficient. If we assume that 900 lb. per square inch of bar is a sufficiently high tangential pressure for a 36 in. commutator, then we can deduce the pressure for any other diameter.

Let p = tangential pressure per square inch, then the total radial pressure per square inch of section will be $2\pi p$, and the outward pressure per cubic inch of copper will be $\frac{2\pi p}{\pi D} = \frac{2p}{D}$. But the resistance to a blow is also aided by the friction between the bars. If, then, k = the coefficient of friction, the frictional resistance = $2kp$.

The total resistance then will be

$$\frac{2p}{D} + 2kp = 2p\left(\frac{1}{D} + k\right).$$

Assuming a coefficient of friction of 0·25 and that 900 lb. per square inch is a sufficient tangential pressure for a 36 in. commutator, the total outward resistance to a blow will be

$$= 1{,}800\left(\frac{1}{36} + \frac{1}{4}\right) = 500 \text{ lb. per cubic inch of copper.}$$

Then

$$2p\left(\frac{1}{D} + 0{\cdot}25\right) = 500,$$

or

$$p = 250\,\frac{4D}{4+D} = 1{,}000\,\frac{D}{D+4}.$$

As the tension in the bolts must balance this pressure with a bolt stress = 12,000 lb. per square inch, we can now obtain the total area of the bolts.

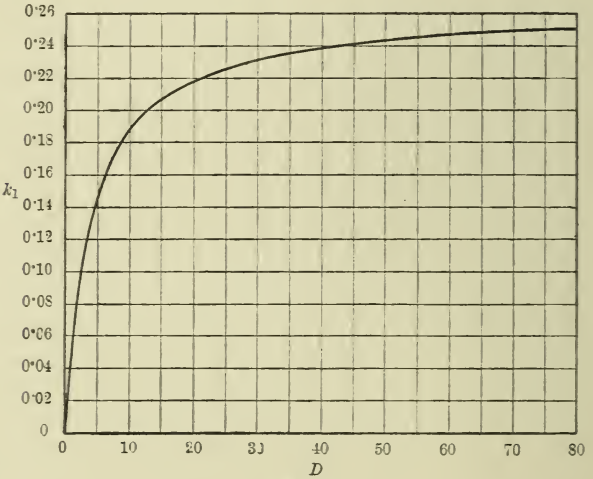


Fig. 8.

The total outward pressure due to initial tension in the commutator will be

$$2\pi B \times 1{,}000\,\frac{D}{D+4} = 6{,}280\,\frac{BD}{D+4},$$

and the total inward pull exerted by the bolts will be

$$12{,}000\text{A} \times 2 \cot \phi.$$

Then $$24{,}000\text{A} \cot \phi = 6{,}280 \frac{\text{BD}}{\text{D}+4}.$$

$$\therefore \quad \text{A} = \frac{6{,}280}{24{,}000 \cot \phi} \times \frac{\text{BD}}{\text{D}+4} = \frac{\text{B}}{3{\cdot}82 \cot \phi} \times \frac{\text{D}}{\text{D}+4};$$

or, if $$k_1 = \frac{\text{D}}{3{\cdot}82(\text{D}+4)},$$

$$\text{A} = \frac{k_1 \text{B}}{\cot \phi}. \quad \ldots \ldots \ldots \ldots \ldots \quad (14)$$

A curve is shown in Fig. 8 giving the values of k_1 for any size of commutator.

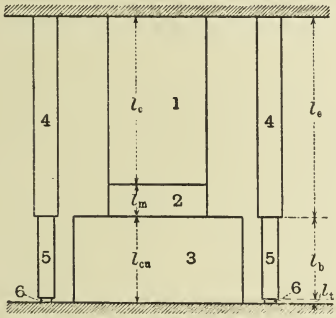


FIG. 9.

Stresses due to Expansion.—It is unnecessary to point out that, when dealing with such a piece of apparatus as a commutator, absolutely exact mathematical expressions for the stresses due to expansion are not necessary and are very inconvenient for practical use; so in the following investigations I shall obtain simple formulæ by making assumptions which will not involve serious errors. In dealing with expansion it will make matters simpler if we resolve the commutator into the diagrammatic form shown in Fig. 9, taking care that our units of length and area are corrected for the change of leverage which this necessitates.

Thus

l_c = Length of copper in the circumference
= Number of bars × mean thickness.

l_m = Length of mica in the circumference
= Number of mica strips × thickness of each.

l_{cu} = Corrected length of copper between "V's"
= $L_1 \times \pi \tan \phi$ (Fig. 6, p. 12).

l_e = Length of end ring
= π × diameter of end ring.

l_b = Corrected length of bolts
= $l_4 \times \pi \tan \phi$ (Fig. 6).

l_t = Corrected length of bolt at bottom of thread
= Length at bottom of thread × $\pi \tan \phi$.

B = Area of bar and area of mica.

B_{cu} = Corrected area of copper annulus
= Area of annulus $\div \pi \tan \phi$.

B_e = Corrected area of end ring = the area of a ring of solid iron which, if placed round the outside of the commutator and the end rings taken away, would be stressed to the same value as the end rings at f_5.

B_b = Corrected area of bolts
= Total area of bolts $\div \pi \tan \phi$.

B_t = Corrected area of bolts at bottom of thread
= Total area at thread $\div \pi \tan \phi$.

If linear coefficient of expansion of copper = 0·0000103
" " iron = 0·0000073

and T° = rise in temperature of copper,
t° = " " iron,
η = difference of expansion in Fig. 8,

then $\eta = 0{\cdot}0000103T^\circ(l_c + l_{cu}) - 0{\cdot}0000073t^\circ(l_e + l_b + l_t)$.

Assuming a rise of temperature of 60 deg. in copper and 25 deg. in iron, then

$$\eta = 0{\cdot}00062(l_c + l_{cu}) - 0{\cdot}000182(l_e + l_b + l_t). \quad . \quad . \quad (15)$$

Now, if in Fig. 9

f_c = Stress in and E_c = Young's modulus of (1)
f_m = " " E_m = " " (2)
f_{cu} = " " E_{cu} = " " (3)
f_e = " " E_e = " " (4)
f_b = " " E_b = " " (5)
f_t = " " E_t = " " (6),

then $f_c B_c = f_m B_m = f_{cu} B_{cu} = f_e B_e = f_b B_b = f_t B_t$, and the difference of expansion must equal compression of 1, 2, and 3 + extension of 4, 5, and 6. That is,

$$\eta = \frac{l_c f_c}{E_c} + \frac{l_m f_m}{E_m} + \frac{l_{cu} f_{cu}}{E_{cu}} + \frac{l_e f_e}{E_e} + \frac{l_b f_b}{E_b} + \frac{l_t f_t}{E_t}.$$

$$\therefore \quad \eta = f_t\left(\frac{l_c B_t}{E_c B_c} + \frac{l_m B_t}{E_m B_m} + \frac{l_{cu} B_t}{E_{cu} B_{cu}} + \frac{l_e B_t}{E_e B_e} + \frac{l_b B_t}{E_b B_b} + \frac{l_t}{E_t}\right).$$

As the length of bolt at bottom of thread is so short, $\frac{l_t}{E_t}$ may be neglected, and

$$f_t = \frac{\eta}{B_t\left(\frac{l_c}{E_c B_c} + \frac{l_m}{E_m B_m} + \frac{l_{cu}}{E_{cu} B_{cu}} + \frac{l_e}{E_e B_e} + \frac{l_b}{E_b B_b}\right)}. \quad . \quad (16)$$

As all the other stresses are in proportion to the bolt stress, we need go no further into the expansion stresses, but simply multiply the initial stresses by $\frac{\text{expansion stress}}{\text{initial bolt stress}}$.

Combined Stresses.—To combine the initial stress with the expansion stress, we simply have to add them together, but we cannot add the centrifugal stress to the sum so obtained. The action of centrifugal force is to *reduce* the stresses due to initial tension and expansion, so we must look for some other method of combining the stresses. If the centrifugal force was sufficient to cause an elongation in the end rings and bolts equal to η, then the expansion stresses would vanish, and if sufficient to cause an elongation $= \frac{\eta\ (\text{initial} + \text{expansion})}{\text{expansion}}$ then both initial and expansion stresses would vanish. Let this ultimate expansion $= \eta_1$. The stress in the bolts to cause an elongation in bolts and end rings $= \eta$ will be $\frac{30 \times 10^6 \eta}{B_t\left(\frac{l_e}{B_e} + \frac{l_b}{B_b}\right)}$ and the stress in the bolts to produce an elongation $= \eta_1$ will be the stress given by above $\times \frac{\eta_1}{\eta}$. Let this stress $= f_f$, and let stress due to initial + expansion $= f_{i+e}$. Then, by means of the diagram in Fig. 10, we can easily obtain the combined stresses. To construct this diagram, mark off on the horizontal line, to any convenient scale, a

length equal to η_1 and erect a perpendicular at this point. Join the origin to where this perpendicular cuts the stress line f_f. Where this "combined" line cut the stress line f_{i+e}, drop a perpendicular and draw the lines marked "centrifugal" and "initial + expansion." Now, for any centrifugal stress the corresponding combined stress may be read by taking the corresponding point in a vertical direction on the line marked "combined."

Of course, this supposes the combined stresses to be within the elastic limits of the materials. All the stresses, with the exception of the pressure on the mica between bars, will be directly proportional to the bolt stress. The pressure on the mica between the bars will be in proportion to the stress

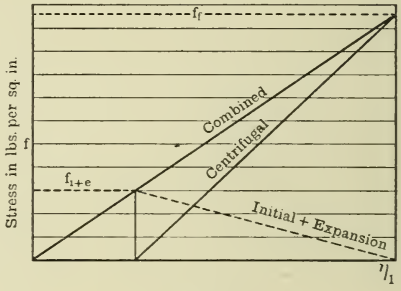


FIG. 10.

obtained from the line marked "initial + compression." The formula for combined stress is

$$Cf = \frac{f_f - f_{i+e}}{f_f} f_c + f_{i+e}, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where Cf = combined stress,

f_c = centrifugal stress.

Buckling of the Bars.—There is certainly a load at which the bars will buckle, but in ordinary cases this load is quite outside the power of the bolts, so we may leave this question entirely out of consideration. I have frequently seen it stated that the bars should not be less than 0·2 in. thick for *mechanical reasons*, but in no case have the reasons been given. There may be particular cases where 0·2 in. really is the

minimum thickness which can be employed, but that does not justify the sweeping statement that bars should not be less than 0·2 in. thick. The only mechanical considerations which really limit the thickness of the bar are: first, the method of manufacturing bars, and second, the method of connecting the armature winding to the commutator. For instance, I have known the following sizes of bar to be used :—

0·14in.	mean thickness in a commutator				30in.	diam. ×	9in.	long
0·125in.	,,	,,	,,	,,	19in.	,, ×	10¼in.	,,
0·105in.	,,	,,	,,	,,	15in.	,, ×	5½in.	,,
0·098in.	,,	,,	,,	,,	8in.	,, ×	3 in.	,,

all without the slightest trouble.

In getting out a new design for a commutator, it is very aggravating to go through the stresses due to centrifugal force,

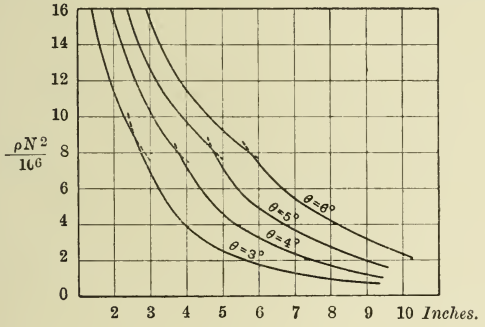


Fig. 11.

initial tension and expansion only to find that some are too high and necessitate some modification in design, and a repetition of all the work with the new dimensions. I find, however, that when the commutator is designed for centrifugal force at a velocity of 3,000 ft. per minute, and high factors of safety, such as are given in the example taken later, are used, little alteration will be necessary to the dimensions. A factor of safety of 2½ to 3 may be allowed for combined stresses at maximum speed.

The curve shown in Fig. 11 will be found useful for obtaining a trial length for the front cut (l).

It is also convenient to know the approximate dimensions of l_1 and the depth of copper below the V's. l_1 may be assumed at $0{\cdot}55l$, and from the actual depth of copper in a large number of commutators I find that the depth k below the "V" $= 0{\cdot}09\mathrm{L} + 0{\cdot}2''$, when L_2 = the maximum length of bar.

The following example will illustrate the use of the formulæ:

Commutator 36 in. diameter, 14 in. long for an eight-pole machine running at 320 revs. per min. $\theta = 3$ deg. $\phi = 30$ deg.

Leads 1 in. wide; mica "V" rings $\frac{1}{10}$ in. thick; mica between bars $\frac{1}{32}$ in. thick; 560 bars.

$$\rho \mathrm{N}^2 = 1{\cdot}75 \times 10^6 \text{ and } \mathrm{P} = 15{\cdot}9.$$

From equation (1) wearing depth of copper should be $1\frac{5}{16}$ in. Then if copper is allowed to wear until $x = \frac{3}{16}$ in., total depth of $x = 1\frac{1}{2}$ in. From Fig. 10 $l = 6\frac{1}{4}$ in., and we will assume $l_1 = 3\frac{1}{8}$ in. and the depth of the copper below the "V" will be 1·46 in. Area of bar will now be $32\frac{1}{4}$ sq. in. = B.

Centrifugal Stresses.—When worn down to $x = \frac{3}{16}$ in.

In equation (2) $$f_1 = \frac{\mathrm{P}l^2(3x + l\tan\theta)}{(x + l\tan\theta)^2};$$

that is, $$f_1 = \frac{15{\cdot}9 \times 6\frac{1}{4}^2(0{\cdot}5625 + 6\frac{1}{4} \times 0{\cdot}0524)}{(0{\cdot}1875 + 0{\cdot}3275)^2}$$

$$= 2{,}100 \text{ lb. per square inch,}$$

and from the equation given in Appendix

$$\delta = k\frac{\mathrm{PL}^2}{\mathrm{E}\tan^2\theta};$$

that is, $$\delta = 0{\cdot}525\frac{15{\cdot}9 \times 6\frac{1}{4}^2}{16 \times 10^6 \times 0{\cdot}0524^2}$$

$$= 0{\cdot}0075 \text{ inches.}$$

In equation (6)

$$f_2 = \frac{6\mathrm{P}\left\{ab\left(l_1 - \frac{a}{2}\right) + \frac{l_1^3\tan\theta}{6} + \frac{x_1(l_1 - a)^2}{2}\right\}}{(l_1\tan\theta + x_1)^2};$$

that is,

$$f_2 = \frac{95{\cdot}4\left\{1{\cdot}663 \times 2{\cdot}625 + \frac{3\frac{1}{8}^3 \times 0{\cdot}0524}{6} + \frac{0{\cdot}3512 \times 2\frac{1}{8}^2}{2}\right\}}{0{\cdot}515^2}$$

$$= 2{,}000 \text{ lb. per square inch.}$$

Remainder of stresses will have a maximum value when the commutator is new :

For stress at f_3, $\frac{D}{L} = 1{\cdot}21$, and from Fig. 5 $\alpha = 16$ deg.

In Fig. 11—R and $R_1 = \frac{1}{8}$ in.

Then

$$\left.\begin{aligned} l_2 &= 1{\cdot}25 \\ l_3 &= 3{\cdot}8 \\ L_1 &= 6{\cdot}6 \\ m &= 1{\cdot}43 \\ d &= 1{\cdot}32 \end{aligned}\right\} \text{Fig. 6.}$$

In equation (9) $f_3 = \dfrac{6PBl_3l_2}{L_1d^2}$;

then $$f_3 = \frac{95{\cdot}4 \times 32{\cdot}25 \times 3{\cdot}8 \times 1{\cdot}25}{6{\cdot}6 \times 1{\cdot}32^2}$$

$$= 1{,}280 \text{ lb. per square inch.}$$

In equation (10) $f_4 = \dfrac{PBl_3}{L_1m}$;

then $$f_4 = \frac{15{\cdot}9 \times 32{\cdot}25 \times 3{\cdot}8}{6{\cdot}6 \times 1{\cdot}43}$$

$$= 205 \text{ lb. per square inch.}$$

Now the total volume of copper is 2,750 cubic inches = V.

And

$$\left.\begin{aligned} h &= 2{\cdot}1 \\ b_1 &= 1{\cdot}55 \\ b_2 &= 0{\cdot}2 \\ l_5 &= 1{\cdot}4 \\ \text{diameter} &= 31{\cdot}1 \end{aligned}\right\} \text{Fig. 7.}$$

Substituting the above values in equation (11) we get

$$f_5 = \frac{3{\cdot}82 \times 2{\cdot}1(15{\cdot}9 \times 2{,}750 \times 3{\cdot}8 \times 1{\cdot}4)}{6{\cdot}6\{1{\cdot}55^3 \times 31{\cdot}1 + 2 \times 2{\cdot}1^3(1{\cdot}55 + 0{\cdot}6)}$$

$$= 1{,}840 \text{ lb. per square inch.}$$

From equation (14) we can obtain the total area of the section of the bolts at the bottom of the thread. From the curve shown in Fig. 9 we obtain $k_1 = 0{\cdot}236$, then the area of the bolts will be $= \dfrac{0{\cdot}236 \times 32{\cdot}25}{1{\cdot}7321} = 4{\cdot}375$ square inches.

At least 14 bolts should be used for this diameter of commutator, so that the nearest size of bolt to give the required

area will be $\frac{3}{4}$ in. diameter. This gives a total area at the bottom of the thread of 4·3 sq. in. The stress in the bolts due to centrifugal force can now be obtained by means of equation (13). Thus, when the commutator is new,

$$f_7 = \frac{15{\cdot}9 \times 2{,}750}{2 \times 1{\cdot}7321 \times 4{\cdot}3} = 2{,}950 \text{ lb. per square inch,}$$

and when the commutator is worn down

$$f_7 = \frac{15{\cdot}9 \times 1{,}200}{2 \times 1{\cdot}7321 \times 4{\cdot}3} = 1{,}280 \text{ lb. per square inch.}$$

Initial Stresses.—New $f_7 = 12{,}000$.

Expansion Stresses.—In Fig. 9

$l_c = 85$	$B_c =$	32·25	when new,	$B_c =$	15·15 worn down.
$l_m = 17{\cdot}5$	$B_m =$	32·25	,,	$B_m =$	15·15 ,,
$l_{cu} = 11{\cdot}9$	$B_{cu} =$	154·0	,,	$B_{cu} =$	93·0 ,,
$l_e = 97{\cdot}5$	$B_e =$	2·37	,,	$B_e =$	2·37 ,,
$l_b = 16{\cdot}6$	$B_b =$	3·44	,,	$B_b =$	3·44 ,,
	$B_l =$	2·37	,,	$B_l =$	2·37 ,,

Substituting the above values in equation (15) we obtain

$$\eta = 0{\cdot}00062(85 + 11{\cdot}9) - 0{\cdot}000182(97{\cdot}5 + 16{\cdot}6) = 0{\cdot}039.$$

With hard-drawn copper bars, C.S. end rings and flaked mica between bars, the following values for Young's modulus may be taken :—

$$E_c \text{ and } E_{cu} = 16 \times 10^6,$$
$$E_e \text{ and } E_b = 30 \times 10^6,$$
$$\text{and } E_m = 0{\cdot}75 \times 10^6.$$

Then, by substituting the numerical values in equation (16), we obtain, when the commutator is new,

$$f_l = \frac{0{\cdot}039}{\frac{2{\cdot}37}{10^6}\left(\frac{85}{16 \times 32{\cdot}25} + \frac{17{\cdot}5}{0{\cdot}75 \times 32{\cdot}25} + \frac{11{\cdot}9}{16 \times 154} + \frac{97{\cdot}5}{30 \times 2{\cdot}37} + \frac{16{\cdot}6}{30 \times 3{\cdot}44}\right)}$$
$$= 6{,}800 \text{ lb. per square inch,}$$

and when the commutator is worn down,

$$f_l = \frac{0{\cdot}039}{\frac{2{\cdot}37}{10^6}\left(\frac{85}{16 \times 15{\cdot}15} + \frac{17{\cdot}5}{0{\cdot}75 \times 15{\cdot}15} + \frac{11{\cdot}9}{16 \times 93} + \frac{97{\cdot}5}{30 \times 2{\cdot}37} + \frac{16{\cdot}6}{30 \times 3{\cdot}44}\right)}$$
$$= 4{,}780 \text{ lb. per square inch.}$$

The initial stresses when worn down will be exactly proportional to the expansion stresses. We can now obtain the combined stresses from the equation given. They will be as follows:—

—	When new.	When worn down.
η	0·039	0·039
η_1	0·108	0·108
f_{i+e}	18,800	13,230
f_f	29,600	29,600
f_c	2,950	1,280
Cf	19,880	13,940

Then the combined stresses at the different points in the commutator will be

Overhanging copper from front cut $kf_1 = 22{,}800$*
" " " rear cut $kf_2 = 22{,}000$†
Copper in "V" ..... $f_3 = 8{,}700$
Mica pressure in "V" ..... $f_4 = 1{,}380$
End ring ..... $f_5 = 12{,}400$
Bolt stress ..... $f_7 = 19{,}880$

* This is slightly high and l should be reduced until kf_1 comes within 20,000. kf is the stress, assuming that the overhanging copper does not deflect, and is really a measure of stiffness.

† This might be somewhat higher without danger.

CHAPTER III.

ASSEMBLING OF "V" RING COMMUTATORS.

The process of building a commutator may be summed up as follows:—

1st. The bars and the mica are built together in a jig with $\frac{1}{8}$ in. Fuller board between the clamp ring and the copper.

2nd. The jig is thoroughly tightened up.

3rd. The bars are heated to a temperature of 200°F. to 250°F. for a period of at least four hours.

4th. The jig is again tightened up and the commutator allowed to cool down.

5th. The end "V's" are now machined out to gauge, the gauge being 0·02 in. larger in diameter than the corresponding female gauge for the mica end rings.

6th. The end rings are put on and tightened up.

7th. The outer jig is released and the surface of commutator turned up.

The successful running of a commutator depends on the workmanship quite as much as on the design, and only the most experienced workmen should be allowed to build up a commutator. Extremely good judgment is required when tightening up the jigs and end rings in order to get the correct tension. Too much pressure is quite as bad as too little, as it often leads to the overstraining of the material and consequent loosening of the bars when run for the first few times.

The form of jig most generally used is that shown in Fig. 12. This consists of a massive outer shell surrounding a thinner split ring. A large number of screws pass through the outer ring and press on the inner one. When the commutator is being pressed together, care should be taken that its circular

shape be retained. It is possible in a jig of this description to press the commutator into whatever shape you please. The form of jig I consider most satisfactory, however, is the hydraulic press. With this form of jig you *know* what pressure you are putting on the commutator; with the jig shown in Fig. 12 you can only guess or, at the best, calculate very roughly. The mica between bars is usually of soft amber mica cut from sheets and built up to the required thickness. Flaked mica, however, of any ordinary quality can be used with good results. The mica is pasted together with shellac to the necessary thickness, then pressed for 12 hours in a mica press, steam heated to a temperature of about 250°F., the pressure being 500 lb. to 1,000 lb. per square inch. It is then taken out and cut into rectangular strips the size of the bar and afterwards milled to thickness. The treatment of the micanite "V" rings is of great importance, and every care

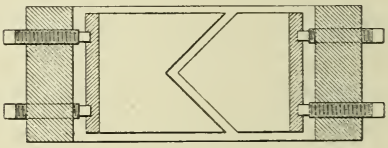


Fig. 12.

should be taken to get them perfectly smooth and of uniform thickness. Any inequalities of the ring are sure to cause trouble and soft spots usually lead to high bars. The mica used should be of the softest quality of amber mica, and so thin that it will stand bending round a very small radius. One form of mould used is that shown in Fig. 13. When moulds are new they should be well waxed, and both male and female portions covered with prepared paper, so that the micanite ring will easily leave the mould.

The mica is then bent into the female portion, each layer being pasted to the previous layer with shellac until a sufficient thickness is obtained. The two portions of the mould are now put together and pressed under a hydraulic press with a pressure of 2,000 lb. per square inch of projected area of mica. I have known lower pressures than 2,000 lb. per square

inch used with satisfactory results; in fact, one writer advises 450 lb. per square inch. The most satisfactory results will be obtained by using the higher pressure, however, as the pressure on the "V" portion of the micanite ring, when built into a commutator, frequently amounts to 1,500 lb. per square inch, and it is advisable to press the micanite rings with a greater pressure than they will receive under running conditions. The micanite ring should be kept under pressure for at least 12 hours, and during that time the mould should be kept at a temperature of 250°F. to 300°F.

It will be necessary to use steel moulds for the higher pressure, though cast iron would be quite good enough for 450 lb. per square inch. All micanite rings should be made in one piece up to 20 in. diameter. Above that diameter it is difficult to get them uniform and they should be made in sections,

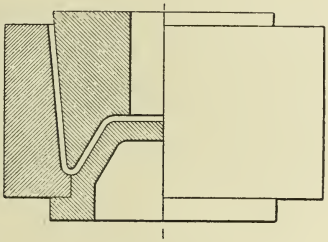


Fig. 13.

each section being half the total thickness required. The whole ring is then built up in two layers with the joints staggered. The object of having Fuller board between the copper and the jig is to allow the insulation between the bars to be tested before the end rings are put on, so that if there are any small particles of copper bridging over any two segments they may be removed.

The best method of heating the commutator is by means of a gas ring. This consists simply of a piece of iron tube, bent to a circle and having small holes on the outer diameter. One end is connected to the gas supply and the other end plugged up. Commutators can be heated very rapidly in this way, and

time is usually a consideration in a modern works. The same remarks apply to the method of heating the mica moulds.

Fig. 14 and Fig. 15 show a type of commutator press made by the West Hydraulic Engineering Co. of Luton. The action is simple and will be readily understood from the illustrations. On the table of the press, which forms a portion of the cast-iron cylinder, is placed a heavy cast-iron bell-mouthed taper ring. The commutator with a temporary clamping band (Fig. 14) is now placed in the bell-mouthed portion of the compression ring, and after the follower and nut are in position

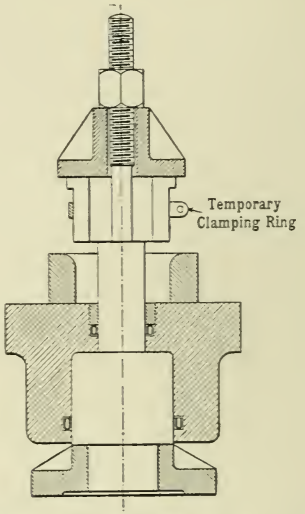


FIG. 14.—WEST HYDRAULIC ENGINEERING CO. "TYPE B" PATENT COMMUTATOR PRESS. (Open.)

pressure is admitted to the annular water space shown; when the commutator is nearly home the temporary clamping ring is removed and the compression ring forced fully home, as shown in Fig. 15.

For long commutators two or more strokes may be necessary, the nut and follower being run down as the table falls. The commutator is now removed to have the "V's" machined, and after the end rings are in position the completed commutator is returned to the press to have the compression ring

removed. This is a speedy way of assembling small commutators and is specially suited for repetition work. The work done by this press is exceedingly accurate, and the commutator turned out truly circular.

To obtain the tangential pressure on the mica between the commutator bars let

W=total pressure exerted by hydraulic ram,

μ=coefficient of friction between copper bars and compression ring,

and F=total radial pressure on the surface of the commutator.

Then $$W = \mu F. \quad \therefore \quad F = \frac{W}{\mu}.$$

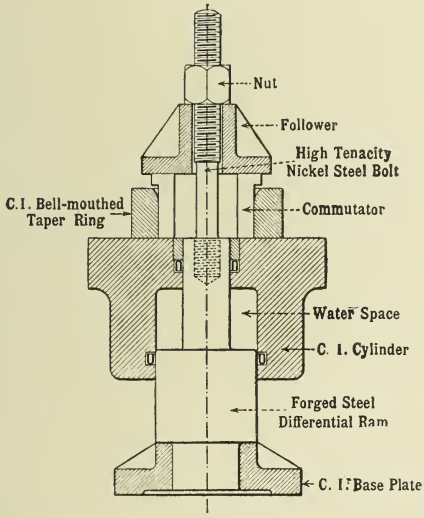


FIG. 15.—WEST HYDRAULIC ENGINEERING CO. "TYPE B" PATENT COMMUTATOR PRESS. (Closed.)

Now, from a well-known theorem in mechanics, the total tangential pressure at any section will be $F/2\pi$ or $W/2\pi\mu$.

Then, if B=the area of the commutator bar, the pressure per square inch on the mica between the bars will be

$$= \frac{W}{2\pi\mu B}.$$

Assuming a coefficient of friction = 0·25, the pressure per square inch of mica will be

$$= 0{\cdot}64\frac{W}{B}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The coefficient of friction will vary considerably with the condition of the surfaces of the compression ring and commutator, and might be anything from 0·12 to 0·3, but 0·25 will be a fair average value.

The preceding only refers to compression rings which have no taper beyond the bell mouth. For tapered rings the following will give the pressure exerted on the commutator.

Let the total taper of compression ring be 1 in η, then for small tapers

$$W = \mu F + \frac{F}{2\eta}, \quad \text{or} \quad W = F\left(\mu + \frac{1}{2\eta}\right),$$

hence $$W = F\left(\frac{2\eta\mu + 1}{2\eta\mu}\right). \quad \therefore F = \frac{2\eta W}{2\eta\mu + 1}.$$

The total tangential pressure will be

$$\frac{\eta W}{\pi(2\eta\mu + 1)},$$

and the pressure per square inch of mica

$$\frac{\eta W}{\pi B(2\eta\mu + 1)}.$$

Assuming, as before, a coefficient of friction of 0·25, the pressure per square inch on mica will be approximately

$$= 0{\cdot}318\frac{W}{B}\left(\frac{\eta}{0{\cdot}5\eta + 1}\right). \quad . \quad . \quad . \quad . \quad . \quad (19)$$

It is evident that the friction in this type of press is an undesirable element, and as this is eliminated in the type of press having radial rams it might be inferred that the radial type could give better results. It is found in practice, however, that this is not so, as it is difficult to get the commutator truly circular in the radial press, due to the pressure being concentrated at a number of points instead of being uniformly distributed all round.

The assembling jig shown in Fig. 16 is one which is used for commutators of large diameter with very successful results. The large flanges enable the jig to be easily chucked in the lathe when machining the end "V's." The bolts in all jigs of

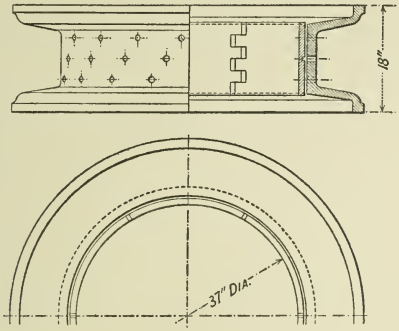


Fig. 16.

this description should be tightened very gradually and the distance between the inner face of the jig and the surface of the commutator frequently calipered all round to ensure that the shape of the finished commutator be truly cylindrical.

CHAPTER IV.

DESIGN OF SHRINK-RING COMMUTATORS.

The shrink-ring type of commutator can be designed on similar lines to the ordinary "V" ring construction, as we have simply replaced the indirect-acting "V" rings by the direct-acting shrink-rings. We will again deal with centrifugal force, initial tension and expansion, in the order named, but the design is very much simplified, as we have a less number of points where the stress need be computed. Assuming that the stress is the same in all the end rings, we may obtain the load on each ring by regarding the bar as a continuous beam, uniformly loaded and supported on level supports. Then, when the shrink-rings exceed two in number, the load on each ring can be obtained by either the deflection method or by Clapeyron's theorem of three moments. The former method is possibly easier of application in this particular case, and in what follows the deflection method will be used.

The simplest case is that shown in Fig. 17, in which only two rings are used. The load on each ring can be determined

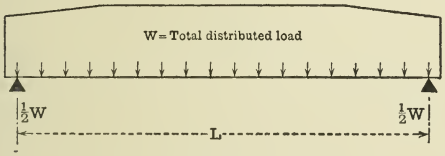


FIG. 17.

statically, and may be taken as approximately half the total load and each ring made half the total section required.

The next case is that shown in Fig. 18. The deflection of a perfectly elastic beam is proportional to the load, within the

limit of elasticity, so we can readily determine the load on the middle ring. As we have assumed that the stress in all the rings shall be of equal value, it follows, since the load on each ring is a constant proportion of the total load, that the supports will always be at one level. Therefore the deflection at the centre of the bar, due to the uniformly distributed load, must equal the opposite deflection caused by the reactance from the centre ring.

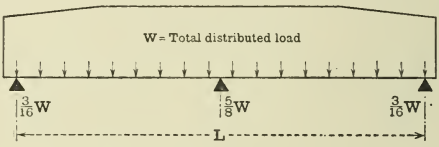


FIG. 18.

If W = the total load uniformly distributed, then the deflection due to that load will be equal to $\frac{5WL^3}{384EI}$ (*see* Appendix), and if W_1 = the load on the middle ring, then the deflection due to W_1 will be equal to $\frac{W_1L^3}{48EI}$ (*see* Appendix). Since both deflections must be equal if all the supports are in one straight line,

$$\frac{5WL^3}{384EI} = \frac{W_1L^3}{48EI};$$

that is,

$$\frac{5W}{384} = \frac{W_1}{48}, \text{ or } W_1 = \tfrac{5}{8}W; \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and, since the whole load is evenly distributed, the remainder of the reactance must be evenly divided between the two end supports. That is, each end support carries $\frac{3}{16}$ of the total load, or 3/16W. The area of the shrink rings, then, should be in the same proportion to the total area required, in order to have the same stress in each ring and keep all supports in one straight line.

In the next case (Fig. 19) the commutator has four shrink rings. This will probably be the maximum number which will be found necessary in practice, and we can apply the same procedure in determining the load on each ring. First of all

we *assume* that the rings are so proportioned that the stress has the same value in each. Then consider a section of the commutator as a continuous beam uniformly loaded and resting on four level supports. It is evident that what is true for one section of the commutator is also true for every section, and so for the whole commutator. In this case the four rings are so spaced that the length of span between each is the same.

The deflection at either of the inside rings, due to the uniformly distributed load W, is equal to $\frac{WL^3}{88{\cdot}5EI}$ (*see* Appendix), and if W_1 equals the load on each of the inner rings, the deflection at either, due to the loads W, is approximately equal to $\frac{W_1L^3}{32{\cdot}4EI}$ (*see* Appendix.)

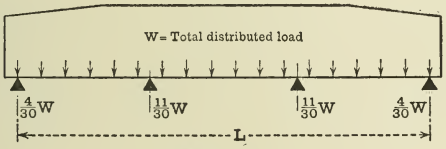


Fig. 19.

Then, since both deflections must be equal if all the supports are in one straight line,

$$\frac{WL^3}{88{\cdot}5EI}=\frac{W_1L^3}{32{\cdot}4EI};$$

that is, $$\frac{W}{88{\cdot}5}=\frac{W_1}{32{\cdot}4}. \quad \therefore\ W_1=\frac{32{\cdot}4W}{88{\cdot}5},$$

or $$W_1=\frac{11}{30}W. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Since the load and supports are symmetrical about the centre of the bar, each of the inside rings carries $\frac{11}{30}$ths of the total load, and the remainder of the load is shared equally by the end rings. That is, each end ring carries $\frac{4}{30}$ths of the total load.

When the commutator is worn down to its limit the conditions are somewhat altered, and it is slightly more difficult to obtain the true load on each ring. The rings are not all

stressed to an equal value, and so the supports no longer remain in one straight line. It will, however, be sufficiently accurate to regard the commutator bar as being split up into the several portions shown in Fig. 20. Portion A can be dealt with as a uniform continuous beam on level supports, just as we have dealt with the bar when it is new, and then we must add the load due to the portions B, C and D to their respective rings. The area of the rings, however, should be in proportion to their respective loads when the bar is new.

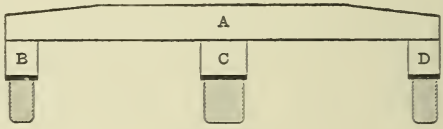


FIG. 20.

In dealing with the stresses in this type of commutator, it will be found convenient to "mass up" the shrink-rings as shown in Fig. 21.

The mean radius of the shrink-ring ρ_1, as shown in Appendix, is equal to

$$\frac{2}{3}\left(\frac{R^3 - r^3}{R^2 - r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

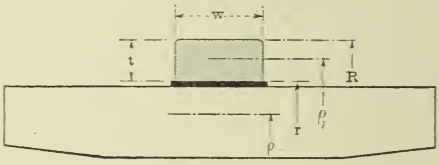


FIG. 21.

The centrifugal pull of one cubic inch of forged steel is $0{\cdot}00000808\rho_1 N^2$. Let this $= P_1$, and let the total volume of the shrink-rings $= V_1$ and the volume of the copper, as before, $= V$. Then the total outward pull of the copper will be PV.

The pressure on the mica under the shrink-ring then will be

$$\frac{PV}{ntw}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

when n = number of bars,

t = thickness at the top of the bar,

and w = width of shrink-rings.

Stress in Shrink-rings.—In addition to the load from the copper the shrink-rings have an additional stress due to their own weight. Thus the total outward pull of copper and iron will be $=PV + P_1V_1$.

To obtain the stress in the ring it will be sufficiently accurate to regard the ring as a thin band whose thickness is small compared with its diameter.

The total outward pressure per inch of the circumference of the ring then will be

$$\frac{PV + P_1V_1}{2\pi r},$$

and from a well-known theorem in mechanics the force tending to burst the ring will be $\frac{2r(PV + P_1V_1)}{2\pi r} = \frac{PV + P_1V_1}{\pi}$. As this force is resisted by any two diametrically opposite sections of the ring the force on any one section will be $\frac{PV + P_1V_1}{2\pi}$, and if the ring, be of the dimensions shown in Fig. 21 the stress in the ring will be

$$\frac{PV + P_1V_1}{2\pi wt} \text{ lb. per square inch.} \quad . \quad (24)$$

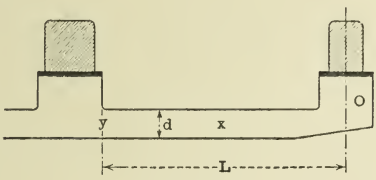


Fig. 22.

Stress at x (Fig. 22).—The stress in the copper will have a maximum value when the bar is worn down to its limit. Although the outer end *o* can hardly be considered to be entirely free, still we shall be on the safe side if we regard it as being so, and simply treat the copper as a uniform beam, uniformly loaded, fixed at one end and supported at the other. The points *x* and *y* are the two places where we will consider the value of the stress.

At x the bending moment is

$$\frac{9PdL^2}{128} \text{ (see Appendix),}$$

and as the modulus of the section at x is $\frac{d^2}{6}$, then the stress at x is

$$\frac{6 \times 9PdL^2}{128d^2} = \frac{27PL^2}{64d} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

At y the bending moment is

$$\frac{PdL^2}{8} \text{ (see Appendix),}$$

and as the modulus of this section is also $\frac{d^2}{6}$, the stress will be

$$\frac{6PdL^2}{8d^2} = \frac{3PL^2}{4d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The deflection of the copper between the shrink-rings must also be kept within reasonable limits, as the modulus of some bars might have a higher or a lower value than the modulus of the others, and the difference of deflection might cause chattering of the brushes.

The deflection at x is

$$\delta = \frac{PdL^4}{187EI} \text{ (see Appendix),}$$

and as

$$I = \frac{d^3}{12},$$

$$\delta = \frac{12PdL^4}{187d^3E} = \frac{PL^4}{15 \cdot 6d^2E} \quad . \quad . \quad . \quad . \quad (27)$$

Initial Stresses.—It is evident that the stresses in the copper are directly proportional to the stress in the shrink-rings. That is, if the commutator is run up to double its normal speed the stresses in the copper and end rings would be four times normal value. If, then, we put an initial stress in the rings, by shrinking them on to the commutator n times the value of the centrifugal stress in the rings, then the stresses and deflections in the copper will be n times the values for centrifugal force.

A commutator of the shrink-ring construction should be made very rigid, as there is no means of tightening up the rings when once the commutator is completed. We should then allow for a greater tangential mica pressure, due to initial

tension in this type than in the "V" ring construction. The exact pressure necessary can only be found from experience, and it is found that in an 18 in. diameter commutator a tangential pressure of 1,200 lb. per square inch gives satisfactory results. Now, by assuming a coefficient of friction, we can deduce the necessary pressure for any other diameter.

With a coefficient of friction of 0·25 the total resistance to a blow is, as before,

$$2P\left(\frac{1}{D}+\frac{1}{4}\right).$$

Then, in an 18 in. commutator with a tangential pressure of 1,200 lb. per square inch the total resistance will be

$$2{,}400\left(\frac{1}{18}+\frac{1}{4}\right)=733 \text{ lb. per cubic inch.}$$

For any commutator of this type, then,

$$2P\left(\frac{1}{D}+\frac{1}{4}\right)=733\ ;$$

that is,
$$2P\left(\frac{4+D}{4D}\right)=733, \text{ or } P=\frac{733}{2}\left(\frac{4D}{4+D}\right),$$

$$\therefore \quad P=1{,}466\left(\frac{D}{D+4}\right). \quad . \quad . \quad . \quad . \quad (28)$$

Assuming, as in the case of the bolts in the "V" ring construction, that this pressure is balanced by the inward pull due to the shrink-rings when the stress in the rings is 12,000 lb. per square inch, we can find the total area of the section of the rings in terms of the area of the bar and diameter of the commutator.

Let B=area of the commutator bar
and A=total area of section of rings ;

then
$$1{,}466B\left(\frac{D}{D+4}\right)=12{,}000A.$$

$$\therefore \quad A=\frac{B}{8{\cdot}2}\left(\frac{D}{D+4}\right). \quad . \quad . \quad . \quad . \quad (29)$$

This gives the total sectional area of the shrink-rings, and it only remains to give to each ring its proper proportion of sectional area.

Thus, in a commutator with two shrink-rings, each ring would have a sectional area of

$$\frac{B}{16\cdot4}\left(\frac{D}{D+4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

In a commutator with three shrink-rings the centre ring would be

$$\frac{5B}{65\cdot6}\left(\frac{D}{D+4}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

while the end rings would have a section

$$\frac{B}{43\cdot7}\left(\frac{D}{D+4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

In a commutator with four shrink-rings each of the centre rings would be made

$$=\frac{B}{22\cdot4}\left(\frac{D}{D+4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

and each of the end rings

$$\frac{B}{61\cdot5}\left(\frac{D}{D+4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

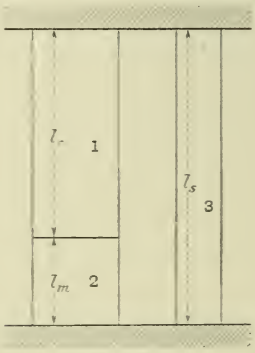


Fig. 23.

Expansion Stresses.—To obtain the stress in the shrink-rings due to expansion we proceed as in the "V" ring construction.

For the sake of clearness, resolve the commutator into its diagrammatic form shown in Fig. 23.

In this case we need make no corrections for change of leverage. Then

l_c=length of copper in the circumference
=number of bars × mean thickness,
l_m=length of mica in the circumference
=number of mica strips × thickness of each,
l_s=length of shrink ring around commutator
=π × mean diameter of ring,
B_c=area of copper bar,
B_m=area of mica segment,
A=total area of section of shrink rings.

Taking again a linear coefficient of expansion for copper =0·0000103, and for iron=0·0000073 :

Then, if T°=rise in temperature of copper,
t°=rise in temperature of iron,
and η=difference of expansion,
then $\eta=0{\cdot}0000103\mathrm{T}^\circ l_c-0{\cdot}0000073tl_s$.

Assuming a rise in temperature of 60°F. in the copper and 30°F. in the iron,

$$\eta=0{\cdot}000618l_c-0{\cdot}000219l_s$$
$$=0{\cdot}000219(2{\cdot}82l_c-l_s). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Now, if in Fig. 23

f_c=Stress in and E_c=Young's modulus of (1),
f_m=Stress in and E_m=Young's modulus of (2),
f_s=Stress in and E_s=Young's modulus of (3),

then $f_cB_c=f_mB_m=f_sA$, and the difference of expansion must equal the compression of 1 and 2 and the extension of 3. That is,

$$\eta=\left(\frac{l_cf_c}{E_c}+\frac{l_mf_m}{E_m}+\frac{l_sf_s}{E_s}\right);$$

and, as $$f_c=\frac{f_sA}{B_c} \text{ and } f_m=\frac{f_sA}{B_m},$$

then $$\eta=f_s\left(\frac{l_cA}{E_cB_c}+\frac{l_mA}{E_mB_m}+\frac{l_s}{E_s}\right).$$

$$\therefore \quad f_s=\frac{\eta.}{\left(\frac{l_cA}{E_cB_c}+\frac{l_mA}{E_mB_m}+\frac{l_s}{E_s}\right)} \quad . \quad . \quad . \quad . \quad (36)$$

All the other stresses will be in proportion to f_s, so we need not repeat any of the work, but simply multiply the centri-

fugal stresses by the ratio of expansion to centrifugal stress in the shrink-rings.

Combined Stresses.—To combine the stresses we must again guard against the mistake of simply adding the several stresses together. We must again resort to a diagram similar to that used for the "V" ring construction, as the centrifugal force in this case also reduces the initial and expansion stresses. We may adopt the same reasoning as before to obtain the combined stress.

If the stress in the shrink-rings due to centrifugal force were so great as to cause an elongation of the rings $\eta_1 = \frac{\eta(\text{initial} + \text{expansion})}{\text{expansion}}$, then both initial and expansion stresses would vanish, and the only stress to consider would be the stress due to centrifugal force.

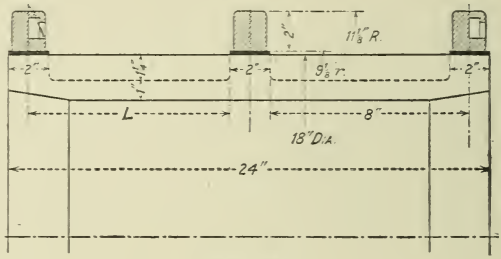


Fig. 24.

The stress in the shrink-rings to cause an elongation $= \eta_1$ will be $= \frac{E_s \times \eta_1}{l_s}$. Let this stress be called f_f, and let the stress due to initial tension + expansion be called f_{i+e}. Then, by means of the diagram in Fig. 10, we can obtain either the combined stress or the initial + expansion stress. The same formula can be used for the combined stress, namely,

$$Cf = \frac{f_f - f_{i+e}}{f_f} f_c + f_{i+e},$$

where Cf = combined stress,

and f_c = centrifugal stress.

To illustrate the use of the formulæ we will take as an example the commutator shown in Fig. 24 for a four-pole generator at 600 revs. per min.

Number of bars 200. Mica between bars 1/32 thick. We can obtain the wearing depth from equation (1), assuming $c=0{\cdot}0005$, $N=600$, and $n=4$.

Then the wearing depth will be $0{\cdot}0005 \times 600 \times 4 = 1{\cdot}2$ in., or, say, $1\frac{1}{4}$ in. Then, as a first approximation, we may take "d" as being $\frac{L}{8}$, or = 1 in.

Then $B = 24 \times 2\frac{1}{4}$ in. = 54 sq. in.

We can now obtain the total area of the shrink-rings from equation (29).

$$A = \frac{54}{8{\cdot}2}\left(\frac{18}{18+4}\right) = 5{\cdot}4 \text{ sq. in.}$$

Then, if the depth is made 2 in., $w=2{\cdot}7$.

Then the centre ring should be 3·375 sq. in.,
and each end ring ,, ,, 1·29 ,,

Centrifugal Stresses.—

$$P = 0\ 00000908 \times 7\tfrac{7}{8} \times 600^2 = 25{\cdot}75,$$

$$V = 54\left(\pi \times 15\tfrac{3}{4} - \frac{200}{32}\right) = 2{,}330 \text{ cubic in.},$$

$$P.V = 60{,}000. \qquad t = 0{\cdot}25 \text{ in.}$$

Then, from equation (23),

$$\textit{Mica Pressure} = \frac{60{,}000}{200 \times 0{\cdot}25 \times 2{\cdot}7} = 445 \text{ lb. per square inch.}$$

Stress in Shrink Rings.—

$$\rho = \tfrac{2}{3}\left(\frac{11\frac{1^3}{8} - 9\frac{1^3}{8}}{11\frac{1^2}{8} - 9\frac{1^2}{8}}\right) = 10{\cdot}2 \text{ in.},$$

$$P_1 = 0{\cdot}00000808 \times 10{\cdot}2 \times 600^2 = 29{\cdot}5,$$

$$V_1 = \pi \times 20\tfrac{1}{4} \times 5{\cdot}4 \text{ sq. in.} = 345 \text{ cubic in.},$$

$$P_1V = 10{,}200.$$

Then, from equation (24),

$$f = \frac{60{,}000 + 10{,}200}{2\pi \times 5{\cdot}4} = 2{,}100.$$

$$L = 8 \text{ in.}$$

Stress at x.—From equation (25)

$$f_x = \frac{27 \times 25\ 75 \times 8^2}{64 \times 1 \text{ in.}} = 700.$$

Stress at y.—From equation (26)

$$f_y = \frac{3 \times 25{\cdot}75 \times 8^2}{4 \times 1} = 1{,}240.$$

Deflection at x.—From equation (27)

$$\delta = \frac{25{\cdot}75 \times 8^4}{15{\cdot}6 \times 1^2 \times 16 \times 10^6} = 0{\cdot}00042.$$

Initial Stresses.—The initial stress in the shrink-rings when the commutator is new will be 12,000 lb. per square inch. The pressure on the mica under the rings will be

$$445 \times \frac{12{,}000}{2{,}100} = 2{,}550 \text{ lb. per square inch.}$$

Expansion Stresses.—

$l_c = 200 \times 0{\cdot}217 = 43{\cdot}4$ in.,
$l_m = 200 \times \frac{1}{32} = 6{\cdot}25$ in.,
$l_s = 20\frac{1}{4}$ in. $\times \pi = 63{\cdot}5$ in.,
$B_c = 54$ sq. in.,
$B_m = 54$ sq. in.,
$A = 5{\cdot}4$ sq. in.;

then $\eta = 0{\cdot}000219(2{\cdot}82 \times 43{\cdot}4 - 63{\cdot}5)$
$= 0{\cdot}000219 \times 59{\cdot}5 = 0{\cdot}0131,$
$E_c = 16 \times 10^6$, $E_m = 10^6$, and $E_s = 28 \times 10^6$;

then
$$f_s = \frac{0{\cdot}0131}{\frac{43{\cdot}4 \times 5{\cdot}4}{16 \times 10^6 \times 54} + \frac{6{\cdot}25 \times 5{\cdot}4}{10^6 \times 54} + \frac{63{\cdot}5}{28 \times 10^6}}$$
$$= \frac{0{\cdot}0131}{0{\cdot}000003167} = 4{,}100.$$

Combined Stresses.—The difference of expansion $= \eta = 0{\cdot}0131$;

then $$\eta_1 = \frac{0{\cdot}0131 \times 16{,}100}{4{,}100} = 0{\cdot}0517,$$

then $$f_t = \frac{28 \times 10^6 \times 0{\cdot}0517}{63{\cdot}5} = 22{,}800,$$

then $$cf = \frac{22{,}800 - 16{,}100}{22{,}800} \times 2{,}100 + 16{,}100$$
$$= 617 + 16{,}100 = 16{,}720.$$

CHAPTER V.

ASSEMBLING OF SHRINK-RING COMMUTATORS.

The process of assembling the shrink-ring type of commutator is, in general, similar to the process of assembling "V" ring commutators. The pressure, however, as I have already pointed out, should be somewhat greater than is found to be sufficient for the "V" ring type, and can be obtained from the equation

$$P = 1{,}466\left(\frac{D}{D+4}\right).$$

If, as before, B=the area of the commutator bar, and W=the total tangential pressure, then W=PB, and as the total radial pressure=$2\pi W$, then the total radial pressure will be

$$2\pi B \times 1{,}466\left(\frac{D}{D+4}\right)$$

$$= 9{,}250B\left(\frac{D}{D+4}\right). \quad . \quad . \quad . \quad . \quad . \quad (37)$$

It should be observed that this expression gives the necessary total radial pressure when the commutator is at atmospheric temperature. As the rings should be shrunk on after the commutator has cooled down, the expression found can be used to give the final squeeze when a radial hydraulic jig is used.

Whilst the commutator is hot much greater pressures than that given by equation (37) can be put on the commutator, but as the commutator cools down, or just before the rings are shrunk on, the pressure should be reduced to that given by the equation so as to avoid overstraining the shrink-rings.

The allowance for shrinkage is an important point, and presents a difficulty of a constructional nature. Interposed between the shrink-rings and the copper is a layer of soft mica. This mica is composed of thin strips built round the commutator and pasted together with shellac to the required thickness—usually $\frac{1}{8}$ in. This mica ring will compress considerably when the ring is shrunk on, but the amount it will compress depends so much on the quality of the mica, &c., that it is difficult to give a value which would suit every case. The best way is to make a preliminary experiment before the rings are shrunk on and base the allowance made on the results of the experiment, keeping in mind that the allowance should be somewhat greater than the experimental value, due to the unevenness of the surface of the commutator. In this way the Author has found that for amber mica, $\frac{1}{8}$ in. thick, which had been previously slightly compressed on the commutator by means of a layer of binding wire, an allowance of 0·02 in. is necessary on the diameter of the shrink-rings. Giving this allowance the indefinite value C, we can obtain an expression for the total allowance in terms of the diameter of the ring. Indeed, taking into consideration the possible error in the compression of the mica, we may regard the ring as being the same diameter as the outside of the commutator without serious error. Then, as the initial stress in the shrink-rings is 12,000 lb. per square inch and D is the diameter of the commmutator,

$$\eta_2 = \frac{12{,}000D}{E} + C,$$

when η_2=allowance for shrinkage and E=modulus of elasticity of ring.

For steel rings E may be taken as 30×10^6, when

$$\eta_2 = 0{\cdot}0004D + C. \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The value of C will be found to vary from 0·008 to 0·025, depending on the mica used and the amount of the compression it receives before the rings are shrunk on.

The rings should be heated until they will easily slip over the mica insulating ring, and before cooling they should be adjusted to their correct position on the commutator. Care should be

taken that the ring is a uniform distance from either end at all points in the circumference of the commutator, otherwise the ring will not run true and will require to be machined after the commutator is completed. It will be found that, in order to get the ring into position easily, it will not be necessary to heat it over 600°F. This will give an increase in diameter of 0·0044D. Thus a 20 in. ring will expand 0·088 in., giving a clearance of about 0·06 in.

The assembling jigs used should be in at least three parts, so that they can be removed from the commutator without interfering with the shrink rings. It will also be found convenient to use three or more such jigs instead of one long one.

Steel binding wire can also be used successfully in assembling this type of commutator. The pull on the binding wire to give the necessary pressure per square inch on the mica, $1{,}466\frac{D}{D+4}$, can be obtained as follows :—

Let F=pull on one strand of binding wire and η=number of strands.

Then the total tangential pull on the commutator will be ηF, and as this must be equal to $1{,}466B\frac{D}{D+4}$,

then
$$F=\frac{1{,}466BD}{\eta(D+4)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

The following table gives the maximum pull allowable on different diameters of binding wire :—

S.W.G.	Safe Pull.		
	Piano Steel.	Steel.	Phosphor Bronze.
	Lb.	Lb.	Lb.
10	1,300	1,100	585
11	1,060	900	480
12	860	730	386
13	670	570	300
14	505	430	227
15	410	340	184
16	325	270	146
17	246	210	110
18	182	155	82
19	126	107	56
20	102	86	46

It should be noted that the above loads can only be allowed for a purpose such as assembling commutators, and they should not be confused with safe *working* loads.

In tightening up the binding wire it will be found convenient to have the wire in a number of separate bands with very long ends, so that while the ends are pulled with the necessary tension the commutator may be rolled backwards and forwards along the binding wire so as to keep each separate turn at the same tension; otherwise the centre portion of the band would not be stressed to the same value as the end turns, due to the large amount of friction between the copper and the binding wire. While the commutator is under pressure it should be heated by a number of gas rings to the temperature given in Chapter III., and after it has cooled down, the rings should be shrunk on and the binding wire removed. The inside of the commutator can now be machined and the assembled bars placed on the shaft or bush. When the end-rings have been tightened the surface of the commutator can be turned and, if necessary, the shrink-rings trued up.

For high-speed commutators successful balancing is a point of great importance. The slightest vibration at high speed will cause a large amount of sparking, so that the commutator *must* be accurately balanced. Even a statical balance is not sufficient, as it is well known that two out-of-balance masses might be diametrically opposite so as to perfectly balance on the levelling bars and yet be at opposite ends of the commutator, so causing large unbalanced forces when the machine is run. To guard against this a dynamical balance should always be obtained before the machine is completed. The most successful way to balance dynamically is to run the commutator on a test shaft supported in two bearings, one of which is allowed to slide a small amount in a horizontal direction perpendicular to the shaft. The movement of the bearing will magnify the effect of out-of-balance forces, and a piece of chalk held close to the commutator will make a mark shortly after the out-of-balance mass has passed. The reason why the chalk mark will not indicate the exact position of the unbalanced, is that the out-of-balance forces take a small period

of time to effect an appreciable movement on the mass of the commutator. The exact amount the chalk mark is behind its correct position can be found by running the commutator in one direction, then in the opposite direction, and taking a point midway between the chalk marks. This will indicate* the correct position for the balance weight. When one end is balanced so as to run true the commutator and shaft should be turned round and the process repeated with the opposite end. Very accurate balance can be obtained by the above method.

* The out-of-balance mass lies on a line passing through a point midway between the chalk marks and the centre line. This method does not indicate on which side of the centre line the unbalanced mass lies. This can only be found by trying the balance weights first on one side, then on the other, and observing whether an improvement is effected in the balance or the reverse.

CHAPTER VI.

DESCRIPTIONS OF VARIOUS TYPES OF COMMUTATORS.

Types of Construction.—In the following pages are shown some modern types of commutator construction. Fig. 25 is one which is largely used by the Westinghouse Company with very satisfactory results. The bush is in one piece with the armature spider, so that relative movement of the armature and commutator is impossible. One disadvantage of this type is that

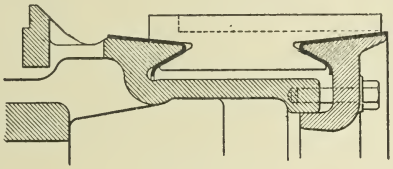


Fig. 25.

the rear "V" must be of cast iron, unless the whole spider is made in cast steel. Another disadvantage is that the studs connecting the bush to the end ring are so short that the expansion of the copper increases the stresses considerably.

The type of construction which was shown in Fig. 1, p. 2, has been adopted by Dick, Kerr & Co. with excellent results. It is a type which has some particularly good points. Both end rings are of cast steel, and in cases where the bars are so long that high expansion stresses would ensue, the bolts are turned down to the same diameter as at the bottom of the thread. This allows of considerable expansion of the copper without unduly increasing the bolt stress. One slight disadvantage of this type is that it is possible for the commutator to have a small movement relatively to the armature. This is

kept extremely small, however, by making the arms of the commutator spider very massive, and no trouble has ever been experienced from this cause.

Fig. 26 is a construction the Author adopted in a design for a very long commutator. It will be noticed that the bolts are carried the full length of the commutator. This allows of considerable expansion of the copper without overstraining

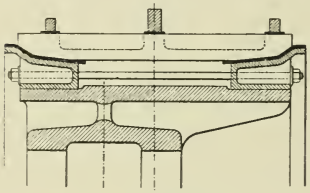


FIG. 26.

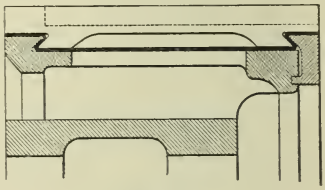


FIG. 27.

the bolts. The longitudinal expansion of the copper was in this case almost $\frac{1}{32}$ in.

Fig. 27.—This type is an example of a bad design, which shows a considerable saving in copper compared with previous designs. It is all right from the manufacturers' point of view to cut out the copper as shown, but it makes an extremely weak commutator. Even if the bar is of sufficient strength to withstand the stresses due to centrifugal force, initial tension, and expansion, when new, the allowable wear will be

FIG. 28.

extremely small. Prof. Goodman describes the practice of loading a column on a side bracket as "barbarous." The same term might as fitly be applied to this practice.

Fig. 28.—In this type of construction the end rings are in sections, so that any of the commutator segments can be easily taken out without disturbing the whole commutator. It seems a big sacrifice of strength for a doubtful advantage. When it

is necessary to renew any of the segments, it is much better to send the whole armature back to the makers, who have all the necessary appliances for such work, and who fully understand the construction. Nothing will shorten the life of a commutator more than unskilful tinkering with it. This type has also the disadvantage that the short length of the bolts, com-

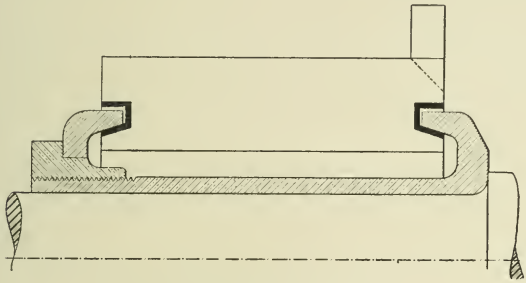


Fig. 29.—Johnson & Phillips.

bined with the additional length of copper between the "V's," is conducive to high-expansion stresses.

Fig. 29 shows a commutator used by Johnson & Phillips, and this commutator has also the disadvantage of the great length of copper between the "V's."

Fig. 30 illustrates the type of construction used on the Tyne

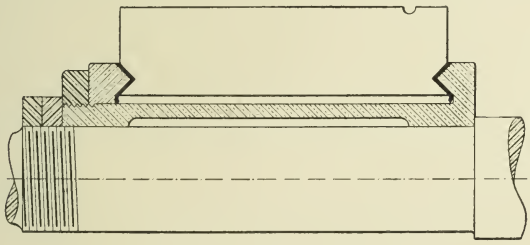


Fig. 30.—Tyne Dynamo.

dynamo and does not present any novel features. This design is by no means economical.

Fig. 31 shows a construction used by Paterson and Cooper, and has a novelty in the form of the end ring; otherwise this commutator is without interest.

Fig. 32 brings us to a most complicated form of commutator. The complications here introduced in the shape of set screws

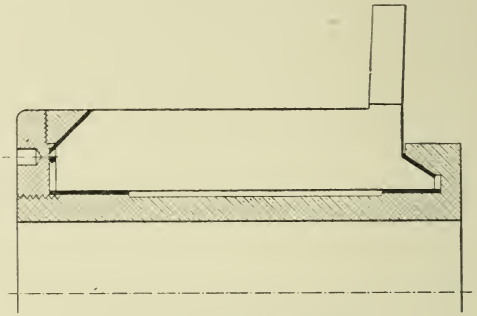


Fig. 31.—Paterson & Cooper.

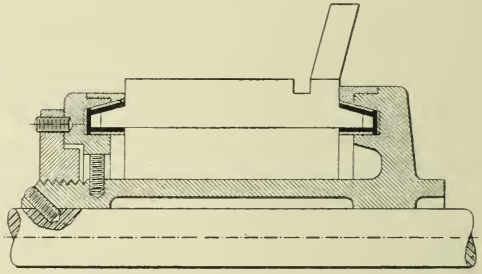


Fig. 32.

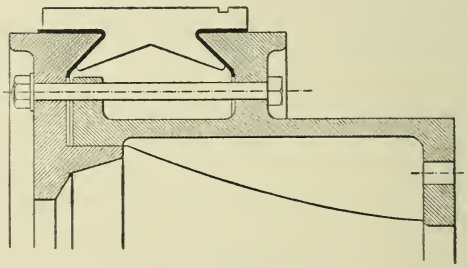


Fig. 33.—Oerlikon Co.

and centring screws are utterly useless. The simplest construction of commutator is usually the best.

Fig. 33 shows a commutator by the Oerlikon Co., in which the copper under the "V's" is cut out. This is bad in principle

and it is doubtful if it is safe even in commutators of very short length. Where it *is* possible to cut out the copper as shown, while also retaining a sufficient factor of safety, it is usually not economical to do so. The cost of cutting out the copper more than counterbalances the saving in material.

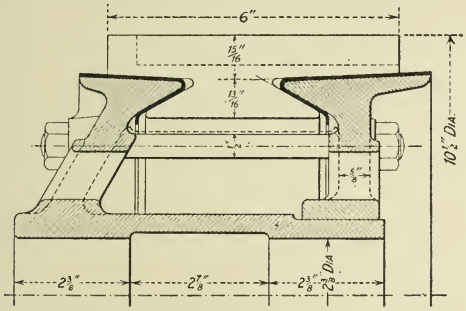


Fig. 34.

The commutator shown in Fig. 34 shows a method of ventilating the interior and at the same time providing an air path to the armature. This is useful in small machines where ventilation is of great importance and the increase of cooling surface on the commutator is considerable. One serious dis-

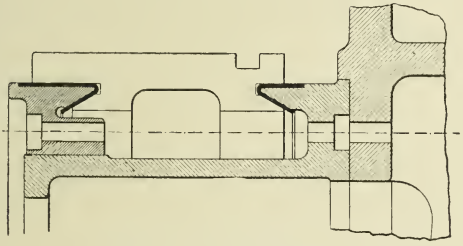


Fig. 35.—Brown Boveri.

advantage to this type is that oil and dirt sometimes collects in the interior of the commutator and cause trouble from short-circuits between the segments. To obviate this disadvantage a thin iron or brass tube can be placed in the inside just over the bolts, as shown in dotted lines in the figure. This, however, whilst it prevents dirt collecting at the bottom of the bars,

reduces the cooling effect somewhat, although at the same time the armature is very efficiently ventilated.

Fig. 35 shows a commutator made by Brown, Boveri and Fig. 36 a construction used by the Oerlikon Co. Both of

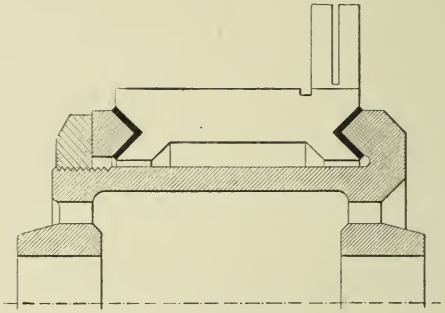


FIG. 36.—OERLIKON CO.

these commutators have the same faults as the commutator shown in Fig. 33.

Fig. 37, by the Oerlikon Co., is a decided improvement on the previous design and indicates a reasonable way of saving

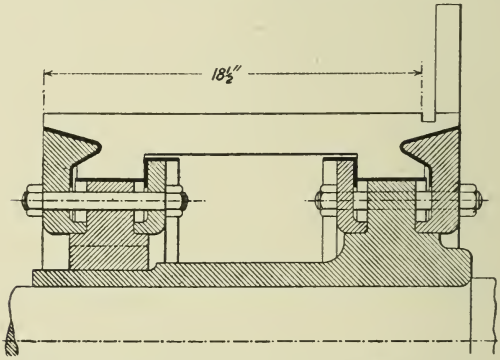


FIG. 37.—OERLIKON CO., FOR LONG COMMUTATORS.

copper in long commutators. Even this design must be kept within limits and the cutting out of copper must not be overdone.

Fig. 38 shows a design used by Dick, Kerr & Co. for small commutators. This design is simple, economical and reliable, the features which all good designs should possess.

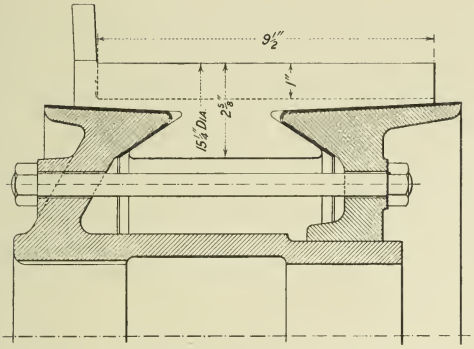


FIG. 38.—DICK, KERR & CO. (SMALL MACHINE).

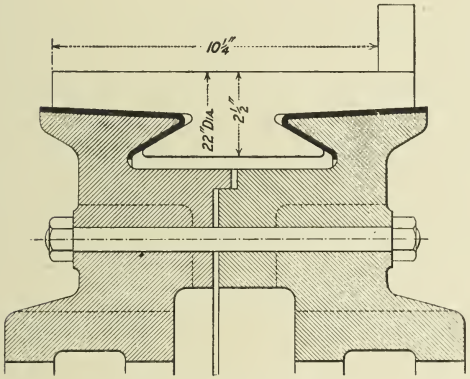


FIG. 39.—BRUCE PEEBLES.

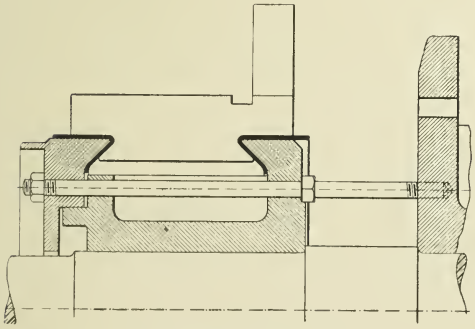


FIG. 40.—KOLBEN & CO.

Fig. 39, a design used by Bruce Peebles, and Fig. 40, a design used by Kolben & Co., are somewhat similar to the last, and show no weak points.

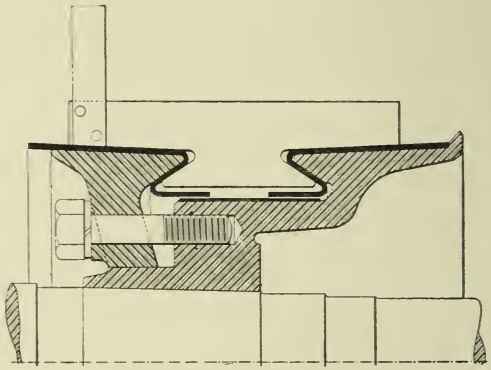


Fig. 41.—Dick, Kerr & Co.

Fig. 41 is used by Dick, Kerr & Co. on railway motors, and Fig. 42 is a design used for the same purpose.

Figs. 43 and 44 are both used by the Oerlikon Co. for high-speed commutators, Fig. 43 being an ordinary commutator with

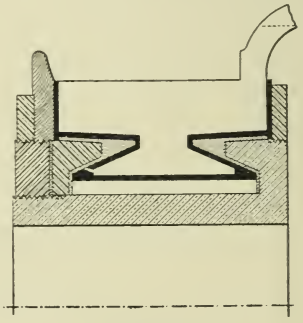


Fig. 42.

an additional shrink-ring in the centre. This design is used for a speed of 1,000 revs. per min.

The last commutator shown (Fig. 45) is an excellent illustration of a commutator for a high-speed generator. This con-

struction was used by the Westinghouse Company for a speed of 2,600 revs. per min. The longitudinal grooves cut in the bars indicate when the commutator should be renewed.

Commutators such as Fig. 27, and indeed *all* commutators, should have some means of indicating when the bars are worn

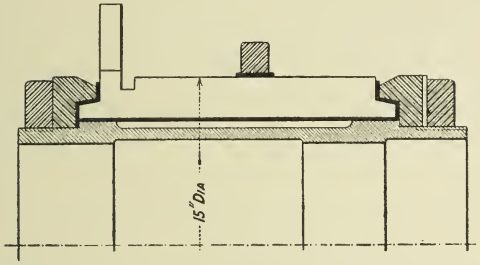


Fig. 43.—Oerlikon Co. (high-speed) 1,000 Revs. per Min.

down as far as they can be with safety. Even a small hole drilled in a few of the bars on the limit line of wear would indicate when it was necessary to renew the commutator. A form of section of bar which is frequently said to increase the strength of the commutator is a ❮ section.

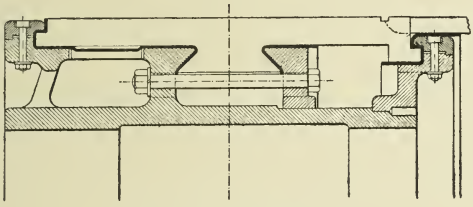


Fig. 44.—Oerlikon Co.

The advantages of this section, however, are more imaginary than real. The section does prevent buckling (if that be considered necessary), and it certainly does prevent any individual bar from being displaced by a blow from a hammer on the outside of the commutator, but the stresses due to centrifugal force, initial tension and expansion are just the same as in a com-

mutator with the ordinary section of bar. These bars are slightly more expensive, and the cost of building a com-

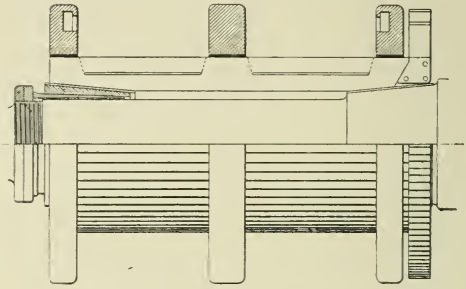


FIG. 45.—WESTINGHOUSE.

mutator with them is greater than with the ordinary type of bar; and I do not think that the slight advantage which this section does possess is worth the additional cost.

CHAPTER VII.

HINTS ON DESIGN.

The best design of commutator is one which is the most economical, consistent with good mechanical proportions, and as copper is the most expensive material used, the section of the bar should be kept as small as possible.

In the "V" type of commutator the angles θ and ϕ used by different manufacturing firms vary to some extent, especially θ. ϕ is usually made about 30 deg. and certainly this angle gives the best ratio of mica pressure to stress in copper, in most cases. The *best* angle θ is more difficult to determine, as it involves the question of economy of material. If, however, we assume that the weight of copper is proportional to the area of the bar, then, expressing the area in terms of tan θ and allowable stress, we have

$$B=\frac{8\tan^2\theta f L_2}{9P}+kL_2-\frac{12{\cdot}4\tan\theta fk}{9P}-\frac{41{\cdot}7\tan^3\theta f^2}{81P^2}+C.$$

Then, for a maximum or minimum value of B, $\frac{dB}{d\tan\theta}$ has zero value—that is,

$$\frac{dB}{d\tan\theta}=\frac{16\tan\theta f L_2}{9P}-\frac{12{\cdot}4fk}{9P}-\frac{125\tan^2\theta f^2}{81P^2}=0$$

$$=16\tan\theta L_2-12{\cdot}4k-\frac{125\tan^2\theta f}{9P}=0$$

$$=\frac{144\tan\theta PL_2}{125f}-\frac{0{\cdot}99kP}{f}-\tan^2\theta=0$$

$$=\tan^2\theta-\frac{1{\cdot}15\tan\theta PL_2}{f}+\frac{0{\cdot}895kP}{f}=0.$$

This is a quadratic equation, which on solving gives

$$\tan\theta=\frac{3PL_2}{5\cdot21f}\pm\sqrt{\left(\frac{3PL_2}{5\cdot21f}\right)^2-\frac{0\cdot895kP}{f}}.$$

Now $$\frac{d^2B}{d\tan^2\theta}=16L_2-\frac{83\cdot1\tan\theta f}{3P},$$

and if we substitute in this expression the value for $\tan\theta$ just found we find that real negative values of $\sqrt{\left(\frac{3PL_2}{5\cdot21f}\right)^2-\frac{\cdot895kP}{f}}$ make the value of the expression positive.

Then for a minimum value of B

$$\tan\theta=\frac{3PL_2}{5\cdot21f}-\sqrt{\left(\frac{3PL_2}{5\cdot21f}\right)^2-\frac{0\cdot895kP}{f}}.$$

$$\therefore\quad\theta=\tan^{-1}\frac{3L_2P}{5\cdot21f}-\sqrt{\left(\frac{3L_2P}{5\cdot21f}\right)^2-\frac{0\cdot895kP}{f}}. \quad . \quad (40)$$

For example, in a commutator 18 in. diam. 23 in. long at 600 revs. per min.

$$P=29\cdot5\ f4{,}500 \text{ and } k=1\cdot25,$$

$$\theta=\tan^{-1}\frac{69\times29\cdot5}{23{,}300}\sqrt{0\cdot00765-0\cdot00733}$$

$$=0\cdot0875-0\cdot018=0\cdot0695.$$

$$\therefore\quad\theta=4^\circ.$$

A small change in the angle, however, makes very little difference in weight and any angle from 3 to 6 deg. is quite suitable for θ.

In order to make this formula applicable when only the diameter and length of the commutator is known, we can substitute for P the outward pull per cubic inch due to initial tension, and the allowable stress for initial tension.

The tangential pressure per square inch is $=1{,}000\frac{D}{D+4}$, and if we multiply by 2π we will get the total radial pressure, and this divided by πD will give the outward pull per cubic inch of copper.

For P, then, we can substitute $\frac{2{,}000}{D+4}$.

The value of f due to initial tension is usually about half the combined stress, so we can substitute for f the value of 12,500, and, as mentioned in Chapter II., k may be assumed at $0{\cdot}09L_2 + 0{\cdot}2$.

$$\text{Then } \theta = \tan^{-1}\left\{\frac{3L_2}{32{\cdot}5(D+4)} - \sqrt{\left(\frac{3L_2}{32{\cdot}5(D+4)}\right)^2 - \frac{0{\cdot}895(0{\cdot}09L_2+0{\cdot}2)}{6\ 25(D+4)}}\right\} \quad (41)$$

As, however, the deflection is frequently the determining factor in the length of the overhanging copper, we will also obtain an expression for the angle θ in terms of the allowable deflection.

From the Appendix, at maximum stress,

$$\delta = 0{\cdot}7\frac{Pl^2}{E\tan^2\theta}.$$

$$\therefore \quad l = \sqrt{\frac{E\tan^2\theta\,\delta}{0{\cdot}7P}}.$$

At a peripheral speed of 3,000 ft. per min.

$$N = \frac{11{,}450}{D} \text{ and } P = \frac{595}{D};$$

$$\text{then} \qquad l = \sqrt{\frac{ED\tan^2\theta\,\delta}{416}} = \frac{\sqrt{ED\tan^2\theta\,\delta}}{20{\cdot}4}.$$

The area of the bar when $E = 16 \times 10^6$ will be

$$-\frac{15\times10^6\tan^3\theta\,\delta}{P} + 4{,}780L\tan^2\theta\sqrt{\frac{\delta}{P}} + kL - 7{,}420k\tan\theta\sqrt{\frac{\delta}{P}} + C,$$

and for maximum or minimum area

$$\frac{dB}{d\tan\theta} = -\frac{45\times10^6\times\delta\tan^2\theta}{P} + 9{,}560L\tan\theta\sqrt{\frac{\delta}{P}} - 7{,}420k\sqrt{\frac{\delta}{P}} = 0$$

$$= \tan^2\theta - \frac{L\tan\theta}{4{,}720}\sqrt{\frac{P}{\delta}} + \frac{k\sqrt{P}}{6{,}080\sqrt{\delta}} = 0,$$

which on solving gives

$$\tan\theta=\frac{L\sqrt{P}}{9{,}440\sqrt{\delta}}\pm\sqrt{\left(\frac{L\sqrt{P}}{9{,}440\sqrt{\delta}}\right)^2-\frac{k\sqrt{P}}{6{,}080\sqrt{\delta}}}.$$

Then, as before, for a minimum value of B,

$$\theta=\tan^{-1}\left\{\frac{L\sqrt{P}}{9{,}440\sqrt{\delta}}-\sqrt{\left(\frac{L\sqrt{P}}{9{,}440\sqrt{\delta}}\right)-\frac{k\sqrt{P}}{6{,}080\sqrt{\delta}}}\right\}.$$

Now, for a peripheral velocity of 3,000 ft. per min.

$$N=\frac{11{,}500}{D}\quad\text{and}\quad P=\frac{595}{D},$$

and if we allow a deflection of 0·0095 we have

$$\theta=\tan^{-1}\left\{\frac{L}{37{\cdot}6\sqrt{D}}-\sqrt{\frac{L^2}{1{,}410D}-\frac{0{\cdot}09L+0{\cdot}2}{24{\cdot}2\sqrt{D}}}\right\}. \quad (42)$$

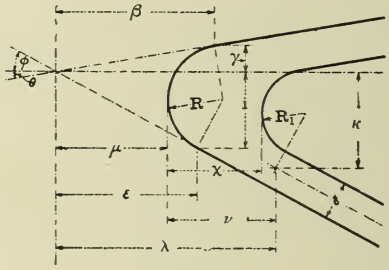


Fig. 46.

The above methods only apply to commutators of ordinary face, and should not be used for commutators with short faces. l should never be more than $\frac{1}{2}$ L, so that, for commutators where the stress limits the overhang, θ should not exceed

$$\tan^{-1}\frac{L}{13{\cdot}5D}, \quad (43)$$

and when the deflection is the determining factor, then θ must not be more than

$$\tan^{-1}\frac{L}{38{\cdot}2\sqrt{D}}. \quad (44)$$

It is not always convenient to draw out a section of a commutator two or three times full size to obtain dimensions such

as m in Fig. 6, p. 12. A more convenient and accurate method is to obtain the general trigonometrical expressions, and then substitute the ratios for the standard angles, so obtaining the necessary dimensions in terms of R, R_1, and t (Fig. 46). Fig. 46 shows the dimensions usually required. Their values are

$$\beta = \frac{\cos\theta}{\tan\frac{\theta+\phi}{2}}R, \quad \gamma = \frac{\sin\theta}{\tan\frac{\theta+\phi}{2}}R,$$

$$\epsilon = \frac{\cos\phi}{\tan\frac{\theta+\phi}{2}}R, \quad \iota = \frac{\sin\phi}{\tan\frac{\theta+\phi}{2}}R,$$

$$\kappa = \frac{\sin\phi}{\tan\frac{\theta+\phi}{2}}R_1 + \left(\frac{\sin\phi}{\tan\frac{\theta+\phi}{2}} - \frac{\cos\phi}{2}\right)t,$$

$$\lambda = \frac{\cos\phi}{\tan\frac{\theta+\phi}{2}}R_1 + \left(\frac{\cos\phi}{\tan\frac{\theta+\phi}{2}} + \frac{\sin\phi}{2}\right)t,$$

$$\mu = \left(\frac{\cos\phi}{\tan\frac{\theta+\phi}{2}} - 1 + \sin\phi\right)R,$$

$$\nu = \frac{\cos\phi}{\tan\frac{\theta+\phi}{2}}R_1 - \left(\frac{\cos\phi}{\tan\frac{\theta+\phi}{2}} - 1 + \sin\phi\right)R + \left(\frac{\cos\phi}{\tan\frac{\theta+\phi}{2}} + \frac{\sin\phi}{2}\right)t,$$

$$\chi = \left(\frac{\cos\phi}{\tan\frac{\theta+\phi}{2}} - 1 + \sin\phi\right)(R_1 - R + t).$$

It is always advisable to have the overhanging portions of the "V" cuts sufficiently long to be consistent with rigidity, as this reduces the length of copper between the "V's" and so also reduces the expansion stresses. It is also "good practice" to turn down the body of the bolts to the same diameter as at the bottom of the thread. This will not be necessary in all commutators, as in commutators with a very short face

the expansion stresses will not be high, but when the length of the commutator exceeds 15 in. the bolts ought to be turned down.

Commutators up to about 20 in. diameter can be successfully made with cast-iron end rings, but above that size steel rings should be used. The saving in weight is appreciable, so that the extra cost is small.

When designing a commutator it is always advisable to make the bolts and end rings to suit the maximum face of commutator, no matter what the face of commutator for the first machine may be. This will keep the number of drawings and patterns down to a minimum.

The same point should also be borne in mind when making the mica moulds.

It is a mistake to make the micanite end rings too thick, as it is very difficult to get the rings of a uniform thickness without soft spots. For small commutators a suitable thickness is $\frac{1}{16}$ in., and for the larger sizes $\frac{1}{10}$ in.

The number of bolts for any diameter of commutator is given by the following approximate formula :—

$$\text{Number of bolts} = \frac{D+4}{3}. \quad . \quad . \quad . \quad . \quad . \quad (45)$$

In the shrink-ring type of commutator the depth of the rings must not be too small. The following will give a suitable depth for any diameter :—

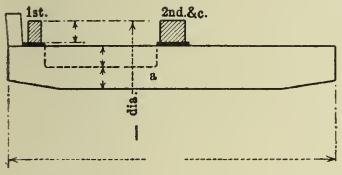
$$\text{Depth} = \frac{D+18}{20}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

The distance between the centres of the rings should be from 9 in. to 14 in. In commutators for high-speed generators some means of balancing must always be provided. Usually the two outer rings are utilised for this purpose, and the section of the ring made as in Fig. 45.

A record of the stresses in a commutator is always useful, and a most convenient method of keeping such a record is by means of a card index.

The form of cards used by the Author is shown in Fig. 47.

____ Dia. ____ Face ____ P. ____ R.P.M. ____ Kw. for ____



Surface Velocity ____ F.P.M.

ρN^2 ____ $\times 10^6$ P = ____

____ Bars. Mean thickness ____ ″

Mica between bars ____ ″ thick.

____ Shrink Rings dimensions as under:—

1st ____ × ____ = ____ sq. inches.

2nd ____ × ____ = ____ ,, ,,

3rd ____ × ____ = ____ ,, ,,

4th ____ × ____ = ____ ,, ,,

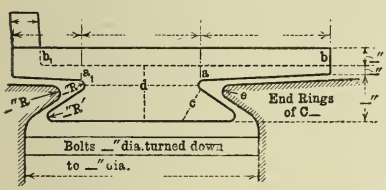
Area of Bar ..... ____ Sq. in.

Total Area of Shrink Rings ____ Sq. in.

		Centrifugal Stresses.	Initial Stresses.	Expansion Stresses. Rise of temp. in Bars ____ °F. Rise of temp. in Rings ____ °F.	Combined Stresses.	Remarks.
a	Bending					
1st	Tension					
2nd	,, ...					
3rd	,, ...					
4th	,, ...					
Mica P under Ring ..						
Tangential Mica Press.						

8″

____ Dia ____ Face ____ P. R.P.M. ____ Kw. for ____



Surface Velocity ____ F.P.M.

ρN^2 ____ $\times 10^6$ P = ____

____ Bars. Mean thickness ____ ″

Mica between bars ____ ″ thick.

,, V Rings ____ ″ ,,

STRESSES AND DEFLECTIONS.

		Centrifugal Stresses.	Initial Stresses.	Expansion Stresses. Rise of temp. in Copper ____ °F. Rise of temp. in Iron ____ °F.	Combined Stresses.	Remarks.
a	Bending					
b	Deflection....					
c	Bending					
a_1	,, ...					
b_1	Deflection....					
d						
e	Tension.....					
Mica Pressure on V. ..						
Bolt Stress						
Tangential Mica Press.						

5″

FIG. 47.—STRESS INDEX CARDS.

Tests on Mica.—In order to obtain the modulus of elasticity of mica under pressure, the Author carried out some tests on a block of built-up mica treated in the same way as the mica segments in a commutator. The curve in Fig. 48 shows the results of the tests.

Curve *a* shows the variation of E during the application of the pressure, curve *b* during the reduction of the pressure, and curve *c* during the reapplication of the pressure. The mica used was flaked amber mica, built together with shellac, and baked to a temperature of 200°F. for 12 hours under a pressure of 400 lb. per square inch.

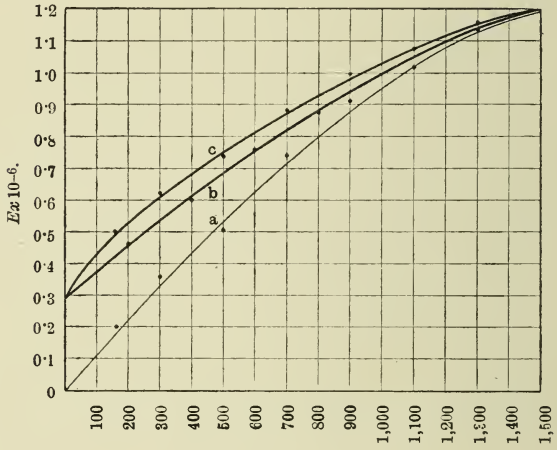


FIG. 48.—MICA TESTS.

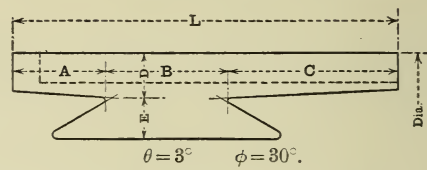
The shellac remaining in the small spaces between the flakes of mica evidently gets squeezed out to some extent and causes a variation of the modulus with repeated application of pressure.

In a certain degree also the mica will break down and cause the same effect. This should be borne in mind when the stresses are computed with the bar worn down to its limit, as before this can take place the mica has been compressed and released many hundred times, and so the centrifugal stresses become of more account when the commutator is worn down. The copper under the "V's," however, must be designed to have the

required factor of safety when the commutator is new, and the deflection of the overhanging copper should not exceed 0·01 when the copper is worn down to its limit.

The table of dimensions given at the end of this chapter will be found to fill the requirements of a good design of commutator, but as it is impossible to make such tables for *every* thickness of bar the dimensions given should be regarded as "preliminary," and the design checked by the methods already given for the particular thickness of bar which is used.

Occasionally the commutator necks have to carry the load due to the armature end connectors, and in such cases the length of the rear overhang may have to be slightly reduced.



$\theta = 3°$ $\phi = 30°$.

L	6 in. diameter.					9 in. diameter.					12 in. diameter.				
	A	B	C	D	E	A	B	C	D	E	A	B	C	D	E
3	$1\frac{1}{8}$	$\frac{3}{4}$	$1\frac{1}{8}$	$\frac{5}{8}$	$\frac{1}{2}$	$1\frac{1}{8}$	$\frac{3}{4}$	$1\frac{1}{8}$	$\frac{5}{8}$	$\frac{1}{2}$	$1\frac{1}{16}$	$\frac{7}{8}$	$1\frac{1}{16}$	$\frac{3}{4}$	$\frac{5}{8}$
4	$1\frac{1}{8}$	$1\frac{1}{4}$	$1\frac{5}{8}$	$\frac{5}{8}$	$\frac{9}{16}$	$1\frac{1}{4}$	$1\frac{5}{8}$	$\frac{5}{8}$	$\frac{9}{16}$		$1\frac{1}{4}$	1	$1\frac{3}{4}$	$\frac{3}{4}$	$\frac{5}{8}$
5	$1\frac{1}{8}$	$1\frac{5}{8}$	$2\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{8}$	$1\frac{3}{8}$	$1\frac{3}{8}$	$2\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{8}$	$1\frac{3}{8}$	$1\frac{1}{4}$	$2\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{8}$
6	$1\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{1}{4}$	$\frac{3}{4}$	$\frac{11}{16}$	$1\frac{1}{2}$	$1\frac{7}{8}$	$2\frac{5}{8}$	$\frac{3}{4}$	$\frac{11}{16}$	$1\frac{1}{2}$	$1\frac{7}{8}$	$2\frac{5}{8}$	$\frac{3}{4}$	$\frac{11}{16}$
7	$1\frac{3}{8}$	$3\frac{3}{8}$	$2\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$1\frac{5}{8}$	$2\frac{3}{8}$	3	$\frac{3}{4}$	$\frac{3}{4}$	$1\frac{3}{4}$	$2\frac{1}{8}$	$3\frac{1}{8}$	$\frac{3}{4}$	$\frac{3}{4}$
8	$1\frac{3}{8}$	$4\frac{3}{8}$	$2\frac{1}{4}$	$\frac{3}{4}$	$\frac{13}{16}$	$1\frac{3}{4}$	$3\frac{1}{4}$	3	$\frac{3}{4}$	$\frac{13}{16}$	$1\frac{7}{8}$	$2\frac{5}{8}$	$3\frac{1}{8}$	$\frac{3}{4}$	$\frac{3}{4}$
9	..	..	..	..	..	$1\frac{7}{8}$	$4\frac{1}{8}$	3	$\frac{7}{8}$	$\frac{7}{8}$	2	$3\frac{1}{8}$	$3\frac{1}{2}$	1	$\frac{7}{8}$
10	..	..	..	..	..	$1\frac{7}{8}$	$5\frac{1}{8}$	3	$\frac{7}{8}$	$\frac{7}{8}$	$2\frac{1}{8}$	$4\frac{3}{8}$	$3\frac{1}{2}$	1	$\frac{7}{8}$
11	..	..	..	..	..	..	..	..	..	..	$2\frac{1}{8}$	$5\frac{1}{4}$	$3\frac{1}{2}$	1	1
12	..	..	..	..	..	..	..	..	..	..	$2\frac{1}{8}$	$6\frac{1}{8}$	$3\frac{1}{2}$	$1\frac{1}{8}$	1

	15 in. diameter.					18 in. diameter.					21 in. diameter.				
3	$1\frac{1}{16}$	$\frac{7}{8}$	$1\frac{1}{16}$	$\frac{3}{4}$	$\frac{5}{8}$	$1\frac{1}{16}$	$\frac{7}{8}$	$1\frac{1}{16}$	$\frac{7}{8}$	$\frac{3}{4}$	$1\frac{1}{16}$	$\frac{7}{8}$	$1\frac{1}{16}$	$\frac{7}{8}$	$\frac{3}{4}$
4	$1\frac{1}{4}$	1	$1\frac{3}{4}$	$\frac{3}{4}$	$\frac{5}{8}$	$1\frac{1}{4}$	1	$1\frac{3}{4}$	$\frac{3}{4}$	$\frac{5}{8}$	$1\frac{1}{8}$	1	$1\frac{7}{8}$	$\frac{7}{8}$	$\frac{7}{8}$
5	$1\frac{3}{8}$	$1\frac{3}{8}$	$2\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{8}$	$1\frac{3}{8}$	$1\frac{3}{8}$	$2\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{8}$	$1\frac{3}{8}$	$1\frac{3}{8}$	$2\frac{1}{4}$	$\frac{7}{8}$	$\frac{7}{8}$
6	$1\frac{5}{8}$	$1\frac{3}{4}$	$2\frac{5}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	$1\frac{5}{8}$	$1\frac{3}{4}$	$2\frac{5}{8}$	1	$\frac{7}{8}$	$1\frac{5}{8}$	$1\frac{3}{4}$	$2\frac{5}{8}$	1	$\frac{7}{8}$
7	$1\frac{3}{4}$	$2\frac{1}{8}$	$3\frac{1}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	$1\frac{7}{8}$	2	$3\frac{1}{8}$	1	$\frac{7}{8}$	$1\frac{3}{4}$	$2\frac{1}{8}$	$3\frac{1}{8}$	1	$\frac{7}{8}$
8	$1\frac{7}{8}$	$2\frac{1}{2}$	$3\frac{5}{8}$	$\frac{7}{8}$	$\frac{7}{8}$	$2\frac{1}{8}$	$2\frac{1}{4}$	$3\frac{5}{8}$	1	1	2	$2\frac{3}{8}$	$3\frac{5}{8}$	1	1
9	2	$3\frac{1}{8}$	$3\frac{5}{8}$	1	1	$2\frac{1}{4}$	$2\frac{3}{4}$	4	$1\frac{1}{8}$	1	$2\frac{3}{8}$	$2\frac{5}{8}$	4	$1\frac{1}{8}$	1
10	$2\frac{1}{8}$	4	$3\frac{7}{8}$	1	1	$2\frac{3}{8}$	$3\frac{3}{8}$	$4\frac{1}{4}$	$1\frac{1}{8}$	1	$2\frac{1}{2}$	3	$4\frac{1}{2}$	$1\frac{1}{8}$	1
11	$2\frac{1}{4}$	$4\frac{7}{8}$	$3\frac{7}{8}$	1	1	$2\frac{1}{2}$	$4\frac{1}{4}$	$4\frac{1}{4}$	$1\frac{1}{8}$	$1\frac{1}{8}$	$2\frac{5}{8}$	$3\frac{3}{4}$	$4\frac{5}{8}$	$1\frac{1}{8}$	$1\frac{1}{8}$
12	$2\frac{3}{8}$	$5\frac{3}{4}$	$3\frac{7}{8}$	$1\frac{1}{8}$	1	$2\frac{5}{8}$	$5\frac{1}{8}$	$4\frac{1}{4}$	$1\frac{1}{4}$	$1\frac{1}{8}$	$2\frac{3}{4}$	$4\frac{5}{8}$	$4\frac{5}{8}$	$1\frac{1}{4}$	$1\frac{1}{8}$
13	..	..	..	..	..	$2\frac{5}{8}$	$6\frac{1}{8}$	$4\frac{1}{4}$	$1\frac{1}{4}$	$1\frac{1}{8}$	$2\frac{7}{8}$	$5\frac{1}{2}$	$4\frac{5}{8}$	$1\frac{1}{4}$	$1\frac{1}{8}$
14	..	..	..	..	..	..	..	..	..	..	$2\frac{7}{8}$	$6\frac{3}{8}$	$4\frac{5}{8}$	$1\frac{1}{4}$	$1\frac{1}{4}$

	24 in. diameter.					30 in. diameter.					36 in. diameter.				
4	$1\frac{1}{2}$	1	$1\frac{1}{2}$	1	$\frac{7}{8}$	$1\frac{1}{2}$	1	$1\frac{1}{2}$	$1\frac{1}{8}$	1	$1\frac{1}{2}$	1	$1\frac{1}{2}$	$1\frac{1}{4}$	$1\frac{1}{8}$
5	$1\frac{5}{8}$	$1\frac{1}{8}$	$2\frac{1}{4}$	1	1	$1\frac{5}{8}$	$1\frac{1}{8}$	$2\frac{1}{4}$	$1\frac{1}{8}$	1	$1\frac{5}{8}$	$1\frac{1}{8}$	$2\frac{1}{4}$	$1\frac{1}{4}$	$1\frac{1}{8}$
6	$1\frac{3}{4}$	$1\frac{5}{8}$	$2\frac{5}{8}$	1	1	$1\frac{3}{4}$	$1\frac{5}{8}$	$2\frac{5}{8}$	$1\frac{1}{8}$	$1\frac{1}{8}$	$1\frac{3}{4}$	$1\frac{5}{8}$	$2\frac{5}{8}$	$1\frac{1}{4}$	$1\frac{1}{4}$
7	2	$1\frac{7}{8}$	$3\frac{1}{8}$	$1\frac{1}{8}$	1	2	$1\frac{7}{8}$	$3\frac{1}{8}$	$1\frac{1}{4}$	$1\frac{1}{8}$	2	$1\frac{7}{8}$	$3\frac{1}{8}$	$1\frac{3}{8}$	$1\frac{1}{4}$
8	$2\frac{1}{8}$	$2\frac{1}{4}$	$3\frac{5}{8}$	$1\frac{1}{8}$	1	$2\frac{1}{4}$	$2\frac{1}{8}$	$3\frac{5}{8}$	$1\frac{1}{4}$	$1\frac{1}{8}$	$2\frac{1}{4}$	$2\frac{1}{8}$	$3\frac{5}{8}$	$1\frac{3}{8}$	$1\frac{1}{4}$
9	$2\frac{1}{2}$	$2\frac{1}{2}$	4	$1\frac{1}{8}$	$1\frac{1}{8}$	$2\frac{1}{2}$	$2\frac{1}{2}$	4	$1\frac{1}{4}$	$1\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{1}{2}$	4	$1\frac{3}{8}$	$1\frac{3}{8}$
10	$2\frac{5}{8}$	$2\frac{7}{8}$	$4\frac{1}{8}$	$1\frac{1}{4}$	$1\frac{1}{8}$	$2\frac{3}{4}$	$2\frac{3}{4}$	$4\frac{1}{2}$	$1\frac{3}{8}$	$1\frac{1}{4}$	$2\frac{3}{4}$	$2\frac{3}{4}$	$4\frac{1}{2}$	$1\frac{1}{2}$	$1\frac{3}{8}$
11	$2\frac{3}{4}$	$3\frac{3}{8}$	$4\frac{7}{8}$	$1\frac{1}{4}$	$1\frac{1}{8}$	$2\frac{7}{8}$	$3\frac{1}{8}$	5	$1\frac{3}{8}$	$1\frac{1}{4}$	$2\frac{7}{8}$	$3\frac{1}{8}$	5	$1\frac{1}{2}$	$1\frac{3}{8}$
12	$2\frac{7}{8}$	$4\frac{1}{4}$	$4\frac{7}{8}$	$1\frac{1}{4}$	$1\frac{1}{4}$	3	$3\frac{5}{8}$	$5\frac{3}{8}$	$1\frac{3}{8}$	$1\frac{3}{8}$	3	$3\frac{5}{8}$	$5\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{1}{2}$
13	3	$5\frac{1}{8}$	$4\frac{7}{8}$	$1\frac{3}{8}$	$1\frac{1}{4}$	$3\frac{1}{8}$	$4\frac{3}{8}$	$5\frac{1}{2}$	$1\frac{1}{2}$	$1\frac{3}{8}$	$3\frac{1}{8}$	4	$5\frac{3}{8}$	$1\frac{5}{8}$	$1\frac{1}{2}$
14	3	$6\frac{1}{8}$	$4\frac{7}{8}$	$1\frac{3}{8}$	$1\frac{1}{4}$	$3\frac{1}{4}$	$5\frac{1}{4}$	$5\frac{1}{2}$	$1\frac{1}{2}$	$1\frac{3}{8}$	$3\frac{1}{4}$	$4\frac{3}{4}$	6	$1\frac{5}{8}$	$1\frac{1}{2}$
15	..	..	..	..	..	$3\frac{3}{8}$	$6\frac{1}{8}$	$5\frac{1}{2}$	$1\frac{1}{2}$	$1\frac{1}{2}$	$3\frac{3}{8}$	$5\frac{5}{8}$	6	$1\frac{5}{8}$	$1\frac{1}{2}$

L	42 in. diameter.					48 in. diameter.					54 in. diameter.				
	A	B	C	D	E	A	B	C	D	E	A	B	C	D	E
5	$1\frac{7}{8}$	$1\frac{1}{4}$	$1\frac{7}{8}$	$1\frac{1}{1}$	$1\frac{1}{8}$	$1\frac{7}{8}$	$1\frac{1}{4}$	$1\frac{7}{8}$	$1\frac{3}{8}$	$1\frac{1}{4}$	$1\frac{7}{8}$	$1\frac{1}{4}$	$1\frac{7}{8}$	$1\frac{3}{8}$	$1\frac{1}{4}$
6	2	$1\frac{3}{8}$	$2\frac{3}{8}$	$1\frac{1}{4}$	$1\frac{1}{4}$	2	$1\frac{3}{8}$	$2\frac{3}{8}$	$1\frac{3}{8}$	$1\frac{3}{8}$	2	$1\frac{3}{8}$	$2\frac{5}{8}$	$1\frac{3}{8}$	$1\frac{3}{8}$
7	$2\frac{1}{4}$	$1\frac{5}{8}$	$3\frac{1}{8}$	$1\frac{3}{8}$	$1\frac{1}{4}$	$2\frac{1}{4}$	$1\frac{5}{8}$	$3\frac{1}{8}$	$1\frac{1}{2}$	$1\frac{3}{8}$	$2\frac{1}{4}$	$1\frac{5}{8}$	$3\frac{1}{8}$	$1\frac{1}{2}$	$1\frac{3}{8}$
8	$2\frac{1}{2}$	$1\frac{7}{8}$	$3\frac{5}{8}$	$1\frac{3}{8}$	$1\frac{1}{4}$	$2\frac{1}{2}$	$1\frac{7}{8}$	$3\frac{5}{8}$	$1\frac{1}{2}$	$1\frac{3}{8}$	$2\frac{1}{2}$	$1\frac{7}{8}$	$3\frac{5}{8}$	$1\frac{1}{2}$	$1\frac{3}{8}$
9	$2\frac{5}{8}$	$2\frac{3}{8}$	4	$1\frac{3}{8}$	$1\frac{3}{8}$	$2\frac{5}{8}$	$2\frac{3}{8}$	4	$1\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{5}{8}$	$2\frac{3}{8}$	4	$1\frac{1}{2}$	$1\frac{1}{2}$
10	$2\frac{3}{4}$	$2\frac{3}{4}$	$4\frac{1}{2}$	$1\frac{1}{2}$	$1\frac{3}{8}$	$2\frac{3}{4}$	$2\frac{3}{4}$	$4\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{1}{2}$	$2\frac{3}{4}$	$2\frac{3}{4}$	$4\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{1}{2}$
11	3	3	5	$1\frac{1}{2}$	$1\frac{3}{8}$	3	3	5	$1\frac{5}{8}$	$1\frac{1}{2}$	3	3	5	$1\frac{5}{8}$	$1\frac{1}{2}$
12	$3\frac{1}{4}$	$3\frac{3}{8}$	$5\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{1}{2}$	$3\frac{1}{4}$	$3\frac{3}{8}$	$5\frac{3}{8}$	$1\frac{5}{8}$	$1\frac{5}{8}$	$3\frac{1}{4}$	$3\frac{3}{8}$	$5\frac{3}{8}$	$1\frac{5}{8}$	$1\frac{5}{8}$
13	$3\frac{3}{8}$	$3\frac{3}{4}$	$5\frac{7}{8}$	$1\frac{5}{8}$	$1\frac{1}{2}$	$3\frac{3}{8}$	$3\frac{3}{4}$	$5\frac{7}{8}$	$1\frac{3}{4}$	$1\frac{5}{8}$	$3\frac{3}{8}$	$3\frac{3}{4}$	$5\frac{7}{8}$	$1\frac{3}{4}$	$1\frac{5}{8}$
14	$3\frac{1}{2}$	$4\frac{1}{8}$	$6\frac{3}{8}$	$1\frac{5}{8}$	$1\frac{5}{8}$	$3\frac{1}{2}$	$4\frac{1}{8}$	$6\frac{3}{8}$	$1\frac{3}{4}$	$1\frac{5}{8}$	$3\frac{1}{2}$	$4\frac{1}{8}$	$6\frac{3}{8}$	$1\frac{3}{4}$	$1\frac{5}{8}$
15	$3\frac{5}{8}$	$4\frac{7}{8}$	$6\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{5}{8}$	$3\frac{3}{4}$	$4\frac{3}{8}$	$6\frac{7}{8}$	$1\frac{3}{4}$	$1\frac{3}{4}$	$3\frac{3}{4}$	$4\frac{3}{8}$	$6\frac{7}{8}$	$1\frac{3}{4}$	$1\frac{3}{4}$
16	$3\frac{3}{4}$	$5\frac{1}{4}$	$6\frac{1}{2}$	$1\frac{7}{8}$	$1\frac{5}{8}$	$3\frac{7}{8}$	$5\frac{1}{8}$	7	$1\frac{7}{8}$	$1\frac{3}{4}$	$3\frac{7}{8}$	$4\frac{7}{8}$	$7\frac{1}{4}$	$1\frac{7}{8}$	$1\frac{3}{4}$
17	..	..	..	..	..	..	..	..	..	..	$3\frac{7}{8}$	$5\frac{3}{4}$	$7\frac{3}{8}$	$1\frac{7}{8}$	$1\frac{3}{4}$

	60 in. diameter.					66 in. diameter.					72 in. diameter.				
6	2	$1\frac{3}{8}$	$2\frac{5}{8}$	$1\frac{1}{2}$	$1\frac{1}{2}$	2	$1\frac{3}{8}$	$2\frac{5}{8}$	$1\frac{1}{2}$	$1\frac{1}{2}$	2	$1\frac{3}{8}$	$2\frac{5}{8}$	$1\frac{1}{2}$	$1\frac{1}{2}$
7	$2\frac{3}{8}$	$1\frac{1}{2}$	$3\frac{1}{8}$	$1\frac{5}{8}$	$1\frac{1}{2}$	$2\frac{3}{8}$	$1\frac{1}{2}$	$3\frac{1}{8}$	$1\frac{5}{8}$	$1\frac{1}{2}$	$2\frac{3}{8}$	$1\frac{1}{2}$	$3\frac{1}{8}$	$1\frac{5}{8}$	$1\frac{1}{2}$
8	$2\frac{1}{2}$	$1\frac{7}{8}$	$3\frac{5}{8}$	$1\frac{5}{8}$	$1\frac{1}{2}$	$2\frac{1}{2}$	$1\frac{7}{8}$	$3\frac{5}{8}$	$1\frac{5}{8}$	$1\frac{1}{2}$	$2\frac{1}{2}$	$1\frac{7}{8}$	$3\frac{5}{8}$	$1\frac{5}{8}$	$1\frac{1}{2}$
9	$2\frac{3}{4}$	$2\frac{1}{4}$	4	$1\frac{5}{8}$	$1\frac{5}{8}$	$2\frac{3}{4}$	$2\frac{1}{4}$	4	$1\frac{5}{8}$	$1\frac{5}{8}$	$2\frac{3}{4}$	$2\frac{1}{4}$	4	$1\frac{5}{8}$	$1\frac{5}{8}$
10	$2\frac{7}{8}$	$2\frac{5}{8}$	$4\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{5}{8}$	$2\frac{7}{8}$	$2\frac{5}{8}$	$4\frac{1}{2}$	$1\frac{3}{4}$	$1\frac{5}{8}$	$2\frac{7}{8}$	$2\frac{5}{8}$	$4\frac{1}{2}$	$1\frac{3}{4}$	$1\frac{5}{8}$
11	$3\frac{1}{8}$	$2\frac{7}{8}$	5	$1\frac{3}{4}$	$1\frac{5}{8}$	$3\frac{1}{8}$	$2\frac{7}{8}$	5	$1\frac{3}{4}$	$1\frac{5}{8}$	$3\frac{1}{8}$	$2\frac{7}{8}$	5	$1\frac{3}{4}$	$1\frac{5}{8}$
12	$3\frac{3}{8}$	$3\frac{1}{4}$	$5\frac{3}{8}$	$1\frac{3}{4}$	$1\frac{3}{4}$	$3\frac{3}{8}$	$3\frac{1}{4}$	$5\frac{3}{8}$	$1\frac{3}{4}$	$1\frac{3}{4}$	$3\frac{3}{8}$	$3\frac{1}{4}$	$5\frac{3}{8}$	$1\frac{3}{4}$	$1\frac{3}{4}$
13	$3\frac{5}{8}$	$3\frac{1}{2}$	$5\frac{7}{8}$	$1\frac{7}{8}$	$1\frac{3}{4}$	$3\frac{5}{8}$	$3\frac{1}{2}$	$5\frac{7}{8}$	$1\frac{7}{8}$	$1\frac{3}{4}$	$3\frac{5}{8}$	$3\frac{1}{2}$	$5\frac{7}{8}$	$1\frac{7}{8}$	$1\frac{3}{4}$
14	$3\frac{7}{8}$	$3\frac{3}{4}$	$6\frac{3}{8}$	$1\frac{7}{8}$	$1\frac{3}{4}$	$3\frac{7}{8}$	$3\frac{3}{4}$	$6\frac{3}{8}$	$1\frac{7}{8}$	$1\frac{3}{4}$	$3\frac{7}{8}$	$3\frac{3}{4}$	$6\frac{3}{8}$	$1\frac{7}{8}$	$1\frac{3}{4}$
15	4	$4\frac{1}{8}$	$6\frac{7}{8}$	$1\frac{7}{8}$	$1\frac{7}{8}$	4	$4\frac{1}{8}$	$6\frac{7}{8}$	$1\frac{7}{8}$	$1\frac{7}{8}$	4	$4\frac{1}{8}$	$6\frac{7}{8}$	$1\frac{7}{8}$	$1\frac{7}{8}$
16	$4\frac{1}{8}$	$4\frac{5}{8}$	$7\frac{1}{8}$	$1\frac{7}{8}$	$1\frac{7}{8}$	$4\frac{1}{8}$	$4\frac{5}{8}$	$7\frac{1}{4}$	$1\frac{7}{8}$	$1\frac{7}{8}$	$4\frac{1}{8}$	$4\frac{5}{8}$	$7\frac{1}{4}$	$1\frac{7}{8}$	$1\frac{7}{8}$
17	$4\frac{1}{8}$	$5\frac{1}{8}$	$7\frac{3}{4}$	2	$1\frac{7}{8}$	$4\frac{1}{8}$	$5\frac{1}{8}$	$7\frac{3}{4}$	2	$1\frac{7}{8}$	$4\frac{1}{8}$	$5\frac{1}{8}$	$7\frac{3}{4}$	2	$1\frac{7}{8}$
18	..	..	..	..	..	$4\frac{1}{8}$	$5\frac{3}{4}$	$8\frac{1}{8}$	2	$1\frac{7}{8}$	$4\frac{1}{8}$	$5\frac{3}{8}$	$8\frac{1}{2}$	2	$1\frac{7}{8}$

	78 in. diameter.					84 in. diameter.					90 in. diameter.				
8	$2\frac{3}{4}$	$1\frac{5}{8}$	$3\frac{5}{8}$	$1\frac{5}{8}$	$1\frac{1}{2}$	$2\frac{3}{4}$	$1\frac{5}{8}$	$3\frac{5}{8}$	$1\frac{5}{8}$	$1\frac{1}{2}$	$2\frac{3}{4}$	$1\frac{5}{8}$	$3\frac{5}{8}$	$1\frac{5}{8}$	$1\frac{1}{2}$
9	$2\frac{7}{8}$	$2\frac{1}{8}$	4	$1\frac{5}{8}$	$1\frac{5}{8}$	$2\frac{7}{8}$	$2\frac{1}{8}$	4	$1\frac{5}{8}$	$1\frac{5}{8}$	$2\frac{7}{8}$	$2\frac{1}{8}$	4	$1\frac{5}{8}$	$1\frac{5}{8}$
10	$3\frac{1}{8}$	$2\frac{3}{8}$	$4\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{5}{8}$	$3\frac{1}{8}$	$2\frac{3}{8}$	$4\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{5}{8}$	$3\frac{1}{8}$	$2\frac{3}{8}$	$4\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{5}{8}$
11	$3\frac{1}{4}$	$2\frac{3}{4}$	5	$1\frac{3}{4}$	$1\frac{5}{8}$	$3\frac{1}{4}$	$2\frac{3}{4}$	5	$1\frac{3}{4}$	$1\frac{5}{8}$	$3\frac{1}{4}$	$2\frac{3}{4}$	5	$1\frac{3}{4}$	$1\frac{5}{8}$
12	$3\frac{1}{2}$	$3\frac{1}{8}$	$5\frac{3}{8}$	$1\frac{3}{4}$	$1\frac{3}{4}$	$3\frac{1}{2}$	$3\frac{1}{8}$	$5\frac{3}{8}$	$1\frac{3}{4}$	$1\frac{3}{4}$	$3\frac{1}{2}$	$3\frac{1}{8}$	$5\frac{3}{8}$	$1\frac{3}{4}$	$1\frac{3}{4}$
13	$3\frac{7}{8}$	$3\frac{1}{4}$	$5\frac{7}{8}$	$1\frac{7}{8}$	$1\frac{3}{4}$	$3\frac{7}{8}$	$3\frac{1}{4}$	$5\frac{7}{8}$	$1\frac{7}{8}$	$1\frac{3}{4}$	$3\frac{7}{8}$	$3\frac{1}{4}$	$5\frac{7}{8}$	$1\frac{7}{8}$	$1\frac{3}{4}$
14	$4\frac{1}{8}$	$3\frac{1}{2}$	$6\frac{3}{8}$	$1\frac{7}{8}$	$1\frac{3}{4}$	$4\frac{1}{8}$	$3\frac{1}{2}$	$6\frac{3}{8}$	$1\frac{7}{8}$	$1\frac{3}{4}$	$4\frac{1}{8}$	$3\frac{1}{2}$	$6\frac{3}{8}$	$1\frac{7}{8}$	$1\frac{3}{4}$
15	$4\frac{3}{8}$	$3\frac{3}{4}$	$6\frac{7}{8}$	$1\frac{7}{8}$	$1\frac{7}{8}$	$4\frac{3}{8}$	$3\frac{3}{4}$	$6\frac{7}{8}$	$1\frac{7}{8}$	$1\frac{7}{8}$	$4\frac{3}{8}$	$3\frac{3}{4}$	$6\frac{7}{8}$	$1\frac{7}{8}$	$1\frac{7}{8}$
16	$4\frac{1}{2}$	$4\frac{1}{4}$	$7\frac{1}{4}$	$1\frac{7}{8}$	$1\frac{7}{8}$	$4\frac{1}{2}$	$4\frac{1}{4}$	$7\frac{1}{4}$	$1\frac{7}{8}$	$1\frac{7}{8}$	$4\frac{1}{2}$	$4\frac{1}{4}$	$7\frac{1}{4}$	$1\frac{7}{8}$	$1\frac{7}{8}$
17	$4\frac{5}{8}$	$4\frac{5}{8}$	$7\frac{3}{4}$	2	$1\frac{7}{8}$	$4\frac{5}{8}$	$4\frac{5}{8}$	$7\frac{3}{4}$	2	$1\frac{7}{8}$	$4\frac{5}{8}$	$4\frac{5}{8}$	$7\frac{3}{4}$	2	$1\frac{7}{8}$
18	$4\frac{3}{4}$	5	$8\frac{1}{4}$	2	2	$4\frac{3}{4}$	5	$8\frac{1}{4}$	2	2	$4\frac{3}{4}$	5	$8\frac{1}{4}$	2	2
19	$4\frac{3}{4}$	$5\frac{1}{2}$	$8\frac{3}{4}$	2	2	$4\frac{3}{4}$	$5\frac{1}{2}$	$8\frac{3}{4}$	2	2	$4\frac{3}{4}$	$5\frac{1}{2}$	$8\frac{3}{4}$	2	2
20	..	..	..	..	..	$4\frac{1}{4}$	$6\frac{1}{4}$	9	2	2	$4\frac{3}{4}$	6	$9\frac{1}{4}$	2	2

APPENDIX.

Centre of Gravity of any Triangle.—In Fig. 49 we will assume that a is the distance of the C.G. from the base of the triangle,

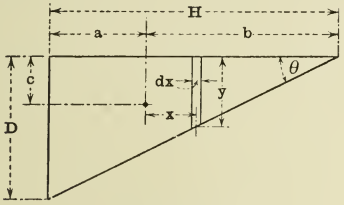


Fig. 49.

and that b is its distance from the apex. Then, taking the sum of the moments of each very small strip dx about a line passing through the C.G., we have for the moments to the right of the line

$$\int_o^b xydx.$$

Now, as $y=\tan\theta(b-x)$, the total moment becomes

$$\int_o^b x\tan\theta(b-x)dx.$$

The total moment of the portion of the triangle on the left of the line will be

$$\int_o^a xydx,$$

and as, in this case, $y=\tan\theta(b+x)$, the total moment will be

$$\int_o^a x\tan\theta(b+x)dx.$$

For a state of equilibrium the above two expressions must be equal—that is,

$$\int_o^b x\tan\theta(b-x)dx=\int_o^a x\tan\theta(b+x)dx$$

$$\equiv\tan\theta\int_o^b(xb-x^2)dx=\tan\theta\int_o^a(xb+x^2)dx,$$

or $$\int_o^b (xb - x^2)dx = \int_o^a (xb + x^2)dx,$$

integrating

$$\left[\frac{x^2b}{2} - \frac{x^3}{3}\right]_o^b = \left[\frac{x^2b}{2} + \frac{x^3}{3}\right]_o^a$$

$$= \frac{b^3}{2} - \frac{b^3}{3} = \frac{a^2b}{2} + \frac{a^3}{3}$$

$$= \frac{b^3}{6} = \frac{a^2}{6}(3b + 2a),$$

or $$b^3 = a^2(3b + 2a).$$

Now, as $b = (H - a)$, then

$$(H - a)^3 = a^2(3H - 3a + 2a)$$

$$= H^3 - 3H^2a + 3Ha^2 - a^3 = 3Ha^2 - a^3$$

$$\equiv H^3 - 3H^2a = 0$$

$$\equiv H^3 = 3H^2a,$$

or $$H = 3a. \qquad \therefore \quad a = \frac{H}{3} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

In the same way we can show that C is equal to D/3.

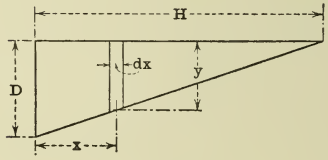


FIG. 50.

Another method is to take the sum of the moments about the base of the triangle and divide by its area. The result will be the distance from the base to the C.G.

In Fig. 50 the moment of any small strip dx about the base is evidently $xydx$.

The total moment then will be

$$\int_o^H xydx.$$

Now $y = \tan\theta(H - x)$, so that the above expression becomes

$$\int_o^H x \tan\theta(H - x)dx$$

$$= \tan\theta \int_o^H (xH - x^2)dx,$$

which on integrating becomes

$$\tan\theta\left[\frac{x^2H}{2}-\frac{x^3}{3}\right]_o^H=\tan\theta\left(\frac{H^3}{2}-\frac{H^3}{2}\right)=\frac{H^3\tan\theta}{6}.$$

Now the area of the triangle is $\frac{H^2\tan\theta}{2}$. $\therefore$ The distance from the C.G. to the base $=\frac{H^3\tan\theta}{6}\div\frac{H^2\tan\theta}{2}=\frac{H}{3}$, as before.

Deflection of a Cantilever due to Centrifugal Force.—In the cantilever shown in Fig. 51 the bending moment at any distance x from the free end (assuming a thickness of 1 in.) is

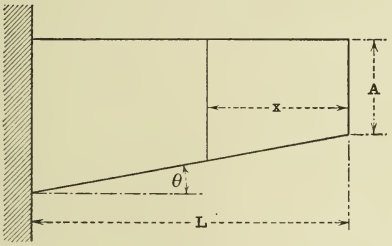


Fig. 51.

$$P\left(\frac{Ax^2}{2}+\frac{x^3\tan\theta}{6}\right).$$

The deflection at the end of the cantilever due to this B.M. will be

$$\frac{P}{EI}\left(\frac{Ax^3}{2}+\frac{x^4\tan\theta}{6}\right);$$

and, as

$$I=\frac{(A+x\tan\theta)^3}{12},$$

the total deflection then will be

$$\delta=\frac{2P}{E}\int_o^L\frac{3Ax^3+x^4\tan\theta}{(A+x\tan\theta)^3}dx.$$

For the sake of clearness we will call $\tan\theta$ B. Then

$$\delta=\frac{2P}{E}\int_o^L\frac{3Ax^3+Bx^4}{(A+Bx)^3}dx.$$

When $\tan\theta$ (or B)$=0$ the above expression becomes

$$\delta=\frac{2P}{E}\int_o^L\frac{3x^3}{A^2}dx=\frac{3PL^4}{2EA^2}. \quad . \quad . \quad . \quad . \quad . \quad (48)$$

and when A=0

$$\delta=\frac{2P}{E}\int_o^L \frac{x}{B^2}dx=\frac{PL^2}{EB^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

To integrate the general expression, however, we must divide out until the numerator of the remaining fraction is of lower degree than the denominator. The terms of the quotient can then be integrated separately and added together.

Proceeding as above we get

$$\delta=\frac{2P}{B^2E}\int_o^L x\,dx-\frac{6A^2P}{BE}\int_o^L \frac{x^2}{(A+Bx)^3}dx-\frac{2A^3P}{B^2E}\int_o^L \frac{x}{(A+Bx)^3}dx.$$

The first integral is evidently $\frac{PL^2}{B^2E}$. . . . . (α)

In the second let $(A+Bx)=Z$,

$$Bdx=dz,$$

and

$$x=\frac{Z-A}{B}.$$

Substituting those values in the integral, we have

$$\frac{6A^2P}{BE}\int\frac{(Z-A)^2}{B^2Z^3}\frac{dz}{B}=\frac{6A^2P}{B^4E}\int\frac{(Z-A)^2}{Z^3}dz,$$

which on integrating becomes

$$\frac{6A^2P}{B^4E}\left[log Z+\frac{2A}{Z}-\frac{A^2}{2Z^2}\right].$$

Substituting $A+Bx$ for Z, we have

$$\frac{6A^2P}{B^4E}\left[log(A+Bx)+\frac{2A}{A+Bx}-\frac{A^2}{2(A+Bx)^2}\right]_o^L$$

$$=\frac{6A^2P}{B^4E}\left\{log\left(1+\frac{BL}{A}\right)+\frac{A(3A+4BL)}{2(A+BL)^2}-\frac{3}{2}\right\} \quad . \quad . \quad (\beta)$$

In the same way we can show that

$$\frac{2A^3P}{B^2E}\int_o^L\frac{x}{(A+Bx)^3}dx$$

is equal to

$$\frac{2A^3P}{B^4E}\left\{\frac{A}{2(A+BL)^2}-\frac{1}{(A+BL)}+\frac{1}{2A}\right\} \quad . \quad . \quad . \quad (\gamma)$$

The total deflection will now be $(\alpha)-(\beta)-(\gamma)$. As the expression is somewhat cumbersome we will proceed immediately to find the deflection at the maximum stress. As we have already noticed, this will be when $A=\frac{BL}{3}$. Substituting this value in equations (α), (β) and (γ), we have

$$(\alpha)=\frac{PL^2}{B^2E},$$

$$(\beta)=\frac{2PL^2}{3B^2E}\left(\log 4+\frac{15}{32}-\frac{3}{2}\right)=\cdot 2366\frac{PL^2}{B^2E},$$

$$(\gamma)=\frac{2PL^3}{27BE}\left(\frac{3}{32BL}-\frac{3}{4BL}+\frac{3}{2BL}\right)=\frac{PL^2}{16B^2E};$$

as $$\delta=(\alpha)-(\beta)-(\gamma),$$

$$\delta=\frac{PL^2}{B^2E}\left(1-\cdot 2366-\frac{1}{16}\right)=\cdot 7009\frac{PL^2}{B^2E}. \quad . \quad . \quad . \quad (50)$$

In the same way we can find a simple expression for the deflection for any value of A.

Let $A=\frac{BL}{C}$, then, substituting this value, we have

$$(\alpha)=\frac{PL^2}{B^2E},$$

$$(\beta)=\frac{6PL^2}{C^2B^2E}\left(\log(1+C)+\frac{\frac{3}{2}+2C}{(1+C)^2}-\frac{3}{2}\right),$$

$$(\gamma)=\frac{2PL^3}{C^3BE}\left(\frac{1}{2CBL\left(\frac{1}{C}+1\right)^2}-\frac{1}{BL\left(\frac{1}{C}+1\right)}+\frac{C}{2BL}\right)$$

$$=\frac{PL^2}{B^2E}\left(\frac{1}{(C+1)^2}\right),$$

$$\delta=\frac{PL^2}{B^2E}\left(1-\frac{6\log(1+C)}{C^2}-\frac{9+12C}{C^2(C+1)^2}+\frac{9}{C^2}-\frac{1}{(C+1)^2}\right),$$

$$\delta=\frac{PL^2}{B^2E}\left(1-\frac{6\log(1+C)-9}{C^2}-\frac{9+12C+C^2}{C^2(C+1)^2}\right), \quad . \quad . \quad (51)$$

when $C=\frac{BL}{A}$.

We now see that the deflection is equal to $k\frac{PL^2}{B^2E}$, where k is a constant depending on the value of C. The curve shown in Fig. 52 gives the values of k corresponding to different values

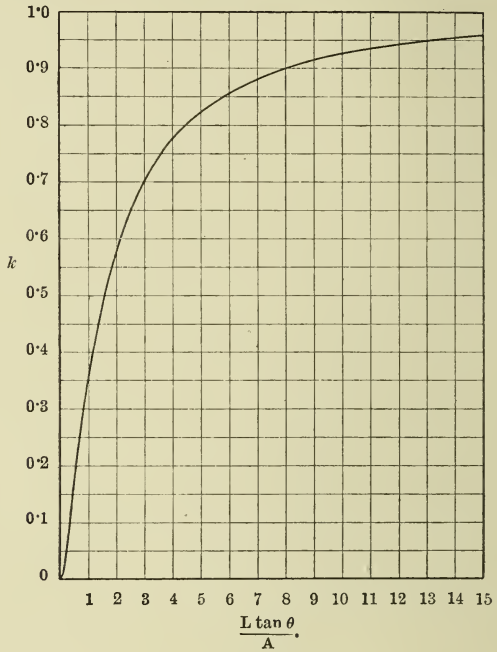


FIG. 52.

of C. It will be observed that 1 is the upper limit of k. This is the case for a cantilever in the form of a triangle—that is, when A=0.

Deflection of a Cantilever Loaded at the Free End.—At any section distant x from the free end (Fig. 53)

$$M = Wx \text{ and } I = \frac{(A + x\tan\theta)^3}{12},$$

when W = the load per inch of breadth of the cantilever.

Then the deflection at the free end due to the moment at that section is $\frac{12Wx^2}{E(A+x\tan\theta)^3}$.

The total deflection then will be

$$\delta = \frac{12W}{E}\int_o^L \frac{x^2}{(A + x\tan\theta)^3}dx\,;$$

when $\tan\theta = 0$ the above becomes

$$\delta = \frac{12W}{EA^3}\int_o^L x^2dx = \frac{4WL^3}{EA^3}, \quad \ldots\ldots\ (52)$$

and when $A = 0$

$$\delta = \frac{12W}{\tan^3\theta E}\int_o^L \frac{1}{x}dx = \frac{12W}{\tan^3\theta E}(\log L - \log 0). \quad \ldots\ (53)$$

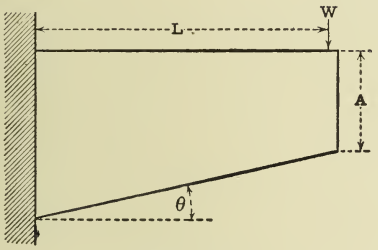


FIG. 53.

To avoid confusion, let $\tan\theta = B$. Then for any value of B and A

$$\delta = \frac{12W}{E}\int_o^L \frac{x^2}{(A + Bx)^3}dx.$$

Let $$A + Bx = Z,$$

then $$Bdx = dz, \quad \text{or} \quad dx = \frac{dz}{B},$$

and $$x = \frac{Z - A}{B}.$$

Substituting, we get

$$\delta = \frac{12W}{EB^3}\int \frac{(Z - A)^2}{Z^3}dz,$$

$$\delta = \frac{12W}{EB^3}\left[\log Z + \frac{2A}{Z} - \frac{A^2}{2Z^2}\right]$$

Substituting for Z its value,

$$\delta = \frac{12W}{EB^3}\left[\log(A+Bx) + \frac{2A}{A+Bx} - \frac{A^2}{2(A+Bx)^2}\right]_o^L,$$

$$\delta = \frac{12W}{EB^3}\left(\log\frac{A+BL}{A} + \frac{2A}{A+BL} - \frac{A^2}{2(A+BL)^2} - \frac{3}{2}\right).$$

Again, let $A = \frac{BL}{C}$, then

$$\delta = \frac{12W}{EB^3}\left(\log(c+1) + \frac{2}{(1+c)} - \frac{1}{2(1+c)^2} - \frac{3}{2}\right). \quad . \quad (54)$$

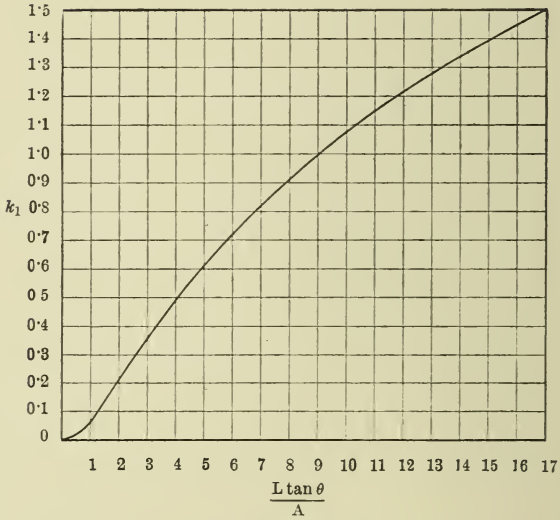


FIG. 54.

It is evident that the above expression also becomes $\delta = k_1 \frac{12W}{EB^3}$, when k_1 is equal to the value inside the brackets. The curve in Fig. 54 gives the values for k_1 corresponding to different values of $\frac{L \tan \theta}{A}$.

Moment of Inertia of a Trapesium about its Base.—The moment of any very small strip dy (Fig. 55) about the base is $y^2x dy$; then the total moment of the trapesium is

$$\int_o^c xy^2 dy.$$

Now $$x = A - \frac{y(A-B)}{C};$$

then $$I = \int_o^c \left(Ay^2 - \frac{A-B}{C} y^3 \right) dy$$

$$= I = \frac{AC^3}{3} - \frac{(A-B)C^3}{4},$$

$$I = \frac{C^3(A+3B)}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

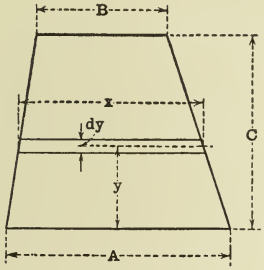


Fig. 55.

Stress in a Section due to Eccentric Loading.—In Fig. 56, if P =

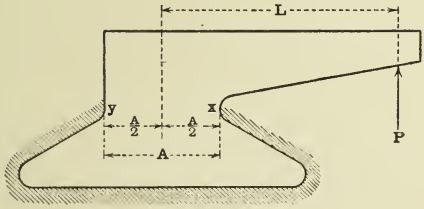


Fig. 56.

the load per inch of material, and f_x be the stress at x, and f_y the stress at y, then the stress due to the load P will be

$$f_x = +\frac{P}{A} \text{ and } f_y = +\frac{P}{A}.$$

The load P, however, also causes a stress due to bending.

$$M=PL \text{ and } Z=\frac{A^2}{6};$$

then $$f_x=+\frac{6PL}{A^2} \quad \text{and} \quad f_y=-\frac{6PL}{A^2}.$$

The total stress at f_x then is

$$\frac{P}{A}+\frac{6PL}{A^2}=\frac{P(A+6L)}{A^2}, \quad \ldots \quad (56)$$

and the total stress at f_y is

$$\frac{P}{A}-\frac{6PL}{A^2}=\frac{P(A-6L)}{A^2}. \quad \ldots \quad (57)$$

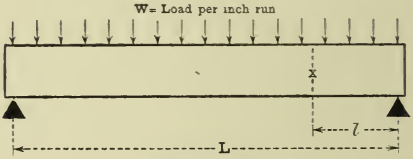


FIG. 57.

Deflection of a Uniform Beam Loaded with a Uniformly Distributed Load.—If w=the load per inch, then at any point x (Fig. 57) the bending moment will be

$$\frac{wLl}{2}-\frac{wl^2}{2},$$

and the deflection due to the bending of the beam at that point will be $$\frac{l}{EI}\left(\frac{wLl}{2}-\frac{wl^2}{2}\right).$$

The total deflection then will be

$$\frac{1}{EI}\int_0^{\frac{L}{2}}\left(\frac{wLl^2}{2}-\frac{wl^3}{2}\right)dl$$

$$=\frac{1}{EI}\left[\frac{wLl^3}{6}-\frac{wl^4}{8}\right]_0^{\frac{L}{2}}$$

$$=\frac{1}{EI}\left(\frac{wL^4}{48}-\frac{wL^4}{128}\right).$$

$$\therefore \quad \delta=\frac{5wL^4}{384EI};$$

or, if the total load $wL = W$, then

$$\delta = \frac{5WL^3}{384EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Deflection of a Uniform Beam with a Single Load at the Centre of the Beam.—In Fig. 58 the bending moment at any

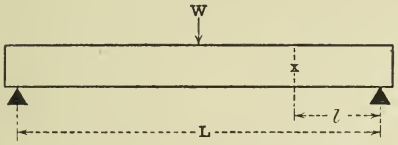


FIG. 58.

point x is $\frac{Wl}{2}$, and the deflection due to the bending at that point will be $\frac{l}{EI}\left(\frac{Wl}{2}\right)$.

The total deflection then will be

$$\frac{1}{EI}\int_0^{\frac{L}{2}} \frac{Wl^2}{2} dl$$

$$= \frac{1}{EI}\left[\frac{Wl^3}{6}\right]_0^{\frac{L}{2}}.$$

$$\therefore \delta = \frac{WL^3}{48EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Bending Moment in a Beam fixed at one end, propped at the other end, and supporting a Uniformly Distributed Load.—Fig. 59

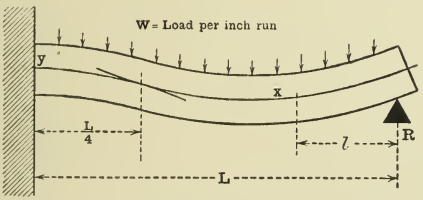


FIG. 59.

represents the case under consideration. We must first of all obtain the reactance on the outer support. To do so

we must obtain expressions for the deflection at the support due to both the load and the reactance.

If w be the load in lbs. per inch, then, neglecting the reactance, the bending moment at any section distant l from the free end will be $wl^2/2$.

The deflection then will be

$$\frac{1}{EI}\int_o^L \frac{wl^3}{2}\,dl=\frac{wL^4}{8EI}.$$

If R be the reactance on the outer support, then the bending moment will be$=Rl$, and the deflection

$$\frac{1}{EI}\int_o^L Rl^2dl=\frac{RL^3}{3EI}.$$

Then, since the two deflections must be equal,

$$\frac{wL^4}{8EI}=\frac{RL}{3EI};$$

that is,

$$R=\frac{3wL}{8}.$$

Now, the bending moment in the cantilever, taking account of the load and the reactance just found, is

$$M=\frac{3wLl}{8}-\frac{wl^2}{2}=\frac{w}{2}\left(\frac{3Ll}{4}-l^2\right),$$

and, for a maximum value,

$$\frac{dM}{dl}=0;$$

that is,

$$\frac{3wL}{8}-wl=0,$$

or

$$\frac{3L}{8}=l.$$

Substituting this value in the general expression for the bending moment, we obtain

$$\text{B.M. at } x=\frac{w}{2}\left(\frac{9L^2}{32}-\frac{9L^2}{64}\right)$$

$$=\frac{9wL^2}{128}. \qquad \ldots\ldots\ldots \quad (59)$$

When $l=L$,

$$\text{B.M. at } y=\frac{w}{2}\left(\frac{3L^2}{4}-L^2\right)$$

$$=-\frac{wL^2}{8}. \quad \ldots\ldots\ldots \quad (60)$$

The point of contrary flexure in this cantilever will be where the bending moment has zero value. That is,

$$\frac{w}{2}\left(\frac{3Ll}{4}-l^2\right)=0,$$

or

$$\frac{3Ll}{4}=l^2.$$

$$\therefore \quad l=\tfrac{3}{4}L.$$

Deflection of the Beam at x.—We can regard the beam as being built up of two separate portions, a cantilever of length $\frac{1}{4}L$ and a freely supported beam of span$=\frac{3}{4}L$. It is evident that the deflection at x is equal to the total deflection of the beam + half the deflection of the cantilever.

Now the bending moment in the cantilever, due to the reactance from the beam, at any section distant l from the point of contrary flexure, is $\frac{3wLl}{8}$, and the bending moment, due to the uniformly distributed load on the cantilever, is $\frac{wl^2}{2}$.

The total B.M. then is $\frac{wl}{2}\left(\frac{3L}{4}+l\right)$,

and the deflection is

$$\frac{1}{EI}\int_0^{\frac{L}{4}}\frac{wl^2}{2}\left(\frac{3L}{4}+l\right)dl$$

$$=\frac{1}{EI}\left[\frac{3wl^3L}{24}+\frac{wl^4}{8}\right]_0^{\frac{L}{4}}$$

$$=\frac{5wL^4}{2048EI}.$$

The B.M. in any section of the beam distant l from the support is

$$\frac{3wLl}{8}-\frac{wl^2}{2}=\frac{wl}{2}\left(\frac{3L}{4}-l\right),$$

and the deflection is

$$\frac{1}{EI}\int^{\frac{3}{8}L}\frac{wl^2}{2}\left(\frac{3L}{4}-l\right)dl$$

$$=\frac{1}{EI}\left[\frac{3wl^3L}{24}-\frac{wl^4}{8}\right]_o^{\frac{3}{8}L}$$

$$=\frac{wL^4}{243EI}\text{ approx.}$$

The total deflection at x then will be

$$\frac{5wL^4}{4096EI}+\frac{wL^4}{243EI}=\frac{wL^4}{187EI} \quad . \quad . \quad . \quad . \quad (61)$$

The maximum deflection is slightly greater than this, but the expression obtained is close enough for practical purposes.

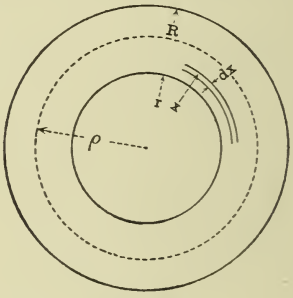


FIG. 60.

Mean Radius of Annular Ring for Calculation of Centrifugal Forces.—The centrifugal pull of any very small strip dx (Fig. 60) is CAx, when C=a constant and A=area of the strip.

The total pull then will be

$$\int_r^R CAx dx.$$

Now, $A=2\pi x$, so the above integral becomes

$$C\int_r^R 2\pi x^2 dx,$$

which, on integrating, becomes

$$\frac{2\pi C}{3}(R^3-r^3).$$

If we divide this expression by C × the total area of the annular disc, we will obtain the desired mean radius.*

The area of the disc $= \pi (R^2 - r^2)$, so that

$$\rho = \frac{2\pi C}{3}(R^3 - r^3) \div \pi C(R^2 - r^2) = \frac{2}{3}\left(\frac{R^3 - r^3}{R^2 - r^2}\right). \quad . \quad (62)$$

When the disc is a solid $r = 0$ and

$$\rho = \frac{2}{3}R. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

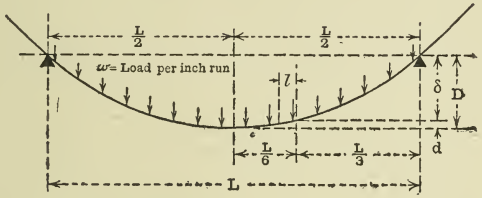


Fig. 61.

Deflection of a Beam at a point one-third of its Length from the Support. Load uniformly distributed.—In Fig. 61

$$\delta = D - d, \; D = \frac{5wL^4}{384EI}.$$

Between $\frac{L}{3}$ and $\frac{L}{2}$,

$$M = \frac{wL}{2}\left(\frac{L}{3} + l\right) - \frac{w}{2}\left(\frac{L}{3} + l\right)^2$$

$$= \frac{wL^2}{6} + \frac{wLl}{2} - \frac{wl^2}{18} - \frac{wl^2}{2} - \frac{wLl}{3}$$

$$= \frac{wL^2}{9} + \frac{wLl}{6} - \frac{wl^2}{2};$$

then $$d = \frac{1}{EI}\int_o^{\frac{L}{6}}\left(\frac{wlL^2}{9} + \frac{wl^2L}{6} - \frac{wl^3}{2}\right)dl,$$

$$= \frac{w}{EI}\left[\frac{l^2L^2}{18} + \frac{l^3L}{18} - \frac{l^4}{8}\right]_o^{\frac{L}{6}},$$

$$= \frac{w}{EI}\left(\frac{L^4}{36 \times 18} + \frac{L^4}{216 \times 18} - \frac{L^4}{1296 \times 8}\right) = \frac{53wL^4}{31104EI}.$$

$$\therefore \quad \delta = \frac{5wL}{384EI} - \frac{53wL^4}{31104EI} = \frac{352wL^4}{31104EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

* This must not be confused with the radius of gyration.

Deflection of a Beam, due to two equal loads uniformly spaced, at the loads.—In Fig. 62

$$\delta = D - d,$$

$$D = \frac{1}{EI}\int_0^{\frac{L}{3}} Rl^2 dl + \frac{1}{EI}\int_0^{\frac{L}{6}} \frac{RL}{3}\left(\frac{L}{3} + l\right) dl,$$

$$= \frac{1}{EI}\left[\frac{Rl^3}{3}\right]_0^{\frac{L}{3}} + \frac{1}{EI}\left[\frac{RLl^2}{9} + \frac{RLl^2}{6}\right]_0^{\frac{L}{6}},$$

$$= \frac{1}{EI}\left(\frac{RL^3}{81} + \frac{RL}{54} + \frac{RL^3}{216}\right) = \frac{23RL^3}{648EI},$$

$$d = \frac{1}{EI}\int_0^{\frac{L}{6}} \frac{RLl}{3}\, dl = \frac{RL}{3EI}\left(\frac{l^2}{2}\right)_0^{\frac{L}{6}} = \frac{RL^3}{216EI}.$$

$$\therefore \quad \delta = \frac{23RL^3}{648EI} - \frac{RL^3}{216EI} = \frac{5RL^3}{162EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

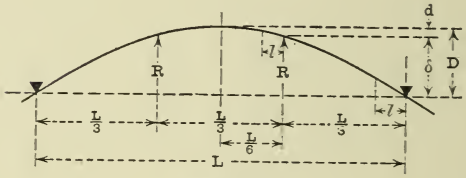


Fig. 62.

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